Nonlinear Analysis 86 (2013) 146–156

Contents lists available at SciVerse ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Multiple solutions for p-Kirchhoff equations in $\mathbb{R}^N$

Caisheng Chen [a,*], Hongxue Song [a,b], Zhonghu Xiu [a,c]

[a] College of Science, Hohai University, Nanjing, 210098, PR China
[b] College of Science, Nanjing University of Posts and Telecommunications, Nanjing, 210023, PR China
[c] Science and Information College, Qingdao Agricultural University, Qingdao, 266109, PR China

ARTICLE INFO

Article history:
Received 29 January 2013
Accepted 30 March 2013
Communicated by S. Carl

MSC:
35B38
35J20
35J92

Keywords:
p-Kirchhoff equation
Mountain Pass Theorem
Ekeland's variational principle
Krasnoselskii's genus theory
Multiple solutions

ABSTRACT

In this paper, we prove the existence of multiple ground-state solutions for the nonhomogeneous p-Kirchhoff elliptic equation

$$M\left(\int_{\mathbb{R}^N}\left(|\nabla u|^p + V(x)|u|^p\right)dx\right)\left(-\Delta_p u + V(x)|u|^{p-2}u\right) = f(x,u) + g(x), \quad \text{in } \mathbb{R}^N, \tag{0.1}$$

where $V(x) \in C(\mathbb{R}^N)$ and $V(x) \to +\infty$ as $|x| \to +\infty$. The nonlinear function $f(x, u)$ is continuous and satisfies some conditions. The solutions will be obtained by the Mountain Pass Theorem, Ekeland's variational principle and Krasnoselskii's genus theory in Struwe (2000) [1].

1. Introduction

In this paper we are interested in the existence of multiple solutions for the following nonlocal p-Kirchhoff elliptic problem of the type

$$\begin{cases} M\left(\int_{\mathbb{R}^N}(|\nabla u|^p + V(x)|u|^p)dx\right)\left(-\Delta_p u + V(x)|u|^{p-2}u\right) = f(x,u) + g(x), & \text{in } \mathbb{R}^N, \\ u(x) \to 0, \quad \text{as } |x| \to +\infty, \end{cases} \tag{1.1}$$

where $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian with $1 < p < N$, and the function $g(x)$ can be seen as a perturbation term.

Such problems are often referred to as being nonlocal because of the presence of the integral over the entire domain Ω. This problem is analogous to the stationary problem of a model introduced by Kirchhoff [2]. More precisely, Kirchhoff proposed a model given by the equation

$$\rho u_{tt} - \left(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L u_x^2 dx\right) u_{xx} = 0, \tag{1.2}$$

where ρ, ρ_0, h, E, L are all positive constants. This equation extends the classical D'Alembert wave equation. For the

* Corresponding author. Tel.: +86 025 83786672; fax: +86 025 83786672.
E-mail address: cshengchen@hhu.edu.cn (C. Chen).

0362-546X/$ – see front matter © 2013 Elsevier Ltd. All rights reserved.
http://dx.doi.org/10.1016/j.na.2013.03.017

bounded domain Ω, the problem

$$\begin{cases} -\left(a + b\int_{\Omega} |\nabla u|^2 dx\right) \Delta u = f(x, u), \quad x \in \Omega, \\ u(x) = 0, \quad x \in \partial\Omega \end{cases} \tag{1.3}$$

is related to the stationary analogue of (1.2). Such nonlocal elliptic problem like (1.3) has received a lot of attention and some important and interesting results have been established in, for example, [3–7]. The study of the Kirchhoff-type equation has already been extended to the case involving the p-Laplacian operator

$$-M\left(\|\nabla u\|_p^p\right)\Delta_p u = f(x, u), \quad x \in \Omega, \tag{1.4}$$

with $M(t) \geq m_0 > 0$ for any $t \geq 0$; see [8–10] and references therein.

Recently, Corrêa and Figueiredo in [10] investigated the multiplicity of the solution to a class of Dirichlet boundary value problems

$$\begin{cases} -\left[M(\|\nabla u\|_p^p)\right]^{p-1} \Delta_p u = f(x, u), \quad x \in \Omega, \\ u = 0, \quad x \in \partial\Omega, \end{cases} \tag{1.5}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$. The functions $M(t), f(x, t)$ are continuous and satisfy the following conditions:

$$a_1 t^\alpha \leq [M(t)]^{p-1} \leq a_2 t^\alpha, \qquad a_3 t^{q-1} \leq f(x, t) \leq a_4 t^{q-1}, \quad x \in \Omega,\ t \geq 0 \tag{1.6}$$

with the constants $a_k > 0$ $(k = 1, 2, 3, 4)$ and $\alpha > q/p,\ q > 1$. Assumption (1.6) implies that $M(0) = 0$, which fails to satisfy the usual assumptions $M(t) \geq m_0 > 0$. A similar study can be also found in [11], in which a possibly degenerate Kirchhoff equation under the homogeneous Dirichlet boundary condition in a bounded domain is studied. The equation is governed by the so-called $p(x)$-polyharmonic operator and the Kirchhoff function is allowed to be zero at zero.

Motivated by these findings, we now extend the analysis to the p-Kirchhoff equation of (1.1) in $\mathbb{R}^N$. We will use the Mountain Pass Theorem, Ekeland's variational principle and Krasnoselskii's genus theory to study the existence of multiple solutions for problem (1.1) with a different nonlinear term $f(x, u)$. Roughly speaking, if $M(t) = t^k, k > 0, m = p(k+1), f(x, u) = h|u|^{r-2}u$, then we will consider the cases: $1 < r \leq m,\ p < m < r$ and $f(x, u) = \lambda h_1|u|^{q-2}u + h_2|u|^{r-2}u$ with $1 < q < m < r$.

Because the domain $\mathbb{R}^N$ is unbounded, the loss of compactness of the Sobolev embedding renders variational techniques more delicate. To preserve this compactness in our problem, we need to impose some conditions on the weight function $V(x)$ as in [12–14].

When $p = 2, k = 0$, by the Mountain Pass Theorem, Toan and Chung [12]studied the existence of (1.1) in an unbounded domain $\Omega \subset \mathbb{R}^N$ under assumptions (H_1) and (H_2) which are stated below. Similar considerations can be found in [13,14] and the references therein.

Throughout this paper, we assume that $M(t) = t^k, k > 0, t \geq 0$ and $V(x)$ is a continuous function satisfying
(H_1) There exists $b_0 > 0$ such that $V(x) \geq b_0$ in $\mathbb{R}^N$. Moreover, $V(x) \to +\infty$ as $|x| \to +\infty$.

Remark 1. For convenience, we assume $b_0 = 1$.

In order to state our main results, we introduce some Sobolev spaces and norms. Let $X = W^{1,p} = W^{1,p}(\mathbb{R}^N)$ be a usual Sobolev space under norm

$$\|u\|_X = \left(\int_{\mathbb{R}^N} (|\nabla u|^p + |u|^p) dx\right)^{1/p}, \quad 1 \leq p < \infty. \tag{1.7}$$

It is well known that the embedding $X \hookrightarrow L^q = L^q(\mathbb{R}^N)$ is continuous and there is a constant $S_q > 0$ such that

$$\|u\|_q \leq S_q \|u\|_X, \quad \forall u \in X \tag{1.8}$$

with $q \in [p, p^*]$. We now consider the following subspace

$$E = \left\{u \in X \,\middle|\, \int_{\mathbb{R}^N} \left(|\nabla u|^p + V(x)|u|^p\right) dx < \infty\right\}. \tag{1.9}$$

Then E is a Banach space with the norm

$$\|u\|_E = \left(\int_{\mathbb{R}^N} \left(|\nabla u|^p + V(x)|u|^p\right) dx\right)^{1/p}. \tag{1.10}$$

Obviously, we have

$$\|u\|_X \leq \|u\|_E, \quad \forall u \in X \tag{1.11}$$