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Grand Lebesgue spaces with respect to measurable functions

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ABSTRACT

Let $1 < p < \infty$. Given $\Omega \subset \mathbb{R}^n$ a measurable set of finite Lebesgue measure, the norm of the grand Lebesgue spaces $L^{p)}(\Omega)$ is given by

$$\|f\|_{L^{p)}(\Omega)} = \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{1}{p-\varepsilon}} \left(\frac{1}{|\Omega|} \int_\Omega |f|^{p-\varepsilon} dx \right)^{\frac{1}{p-\varepsilon}}.$$

In this paper we consider the norm $\|f\|_{L^{p),\delta}(\Omega)}$ obtained replacing $\varepsilon^{\frac{1}{p-\varepsilon}}$ by a generic nonnegative measurable function $\delta(\varepsilon)$. We find necessary and sufficient conditions on δ in order to get a functional equivalent to a Banach function norm, and we determine the "interesting" class $\mathcal{B}_p$ of functions δ, with the property that every generalized function norm is equivalent to a function norm built with $\delta \in \mathcal{B}_p$. We then define the $L^{p),\delta}(\Omega)$ spaces, prove some embedding results and conclude with the proof of the generalized Hardy inequality.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a measurable set of Lebesgue measure $|\Omega| < +\infty$. In 1992 Iwaniec and Sbordone [1] studied the integrability properties of the Jacobian determinant, and introduced the grand Lebesgue space $L^{n)}(\Omega)$ as a space such that

$$|Df| \in L^{n)}(\Omega) \Rightarrow |Jf| \in L^1_{\text{loc}}(\Omega)$$

for all Sobolev mappings $f : \Omega \to \mathbb{R}^n, f = (f^1, \ldots, f^n)$.

Since then the grand Lebesgue spaces play an important role in PDEs theory (see e.g. [2–8]) and in Function Spaces Theory (see e.g. [9–12] and references therein). It turns out that such spaces are Banach Function Spaces in the sense of [13]: namely (here and in the following we will use the letter p instead of n, assuming $1 < p < \infty$)

$$L^{p)}(\Omega) = \left\{ f \in \mathcal{M}_0 : \|f\|_{p)} = \rho(|f|) = \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{1}{p-\varepsilon}} \left(\frac{1}{|\Omega|} \int_\Omega |f|^{p-\varepsilon} dx \right)^{\frac{1}{p-\varepsilon}} < +\infty \right\},$$

where $\mathcal{M}_0$ is the set of all real valued measurable functions on Ω, and, denoting by $\mathcal{M}_0^+$ the subset of $\mathcal{M}_0$ of the nonnegative functions, $\rho : \mathcal{M}_0^+ \to [0, +\infty]$ is such that for all $f, g, f_n (n = 1, 2, 3, \ldots)$ in $\mathcal{M}_0^+$, for all constants $\lambda \geq 0$, and for all measurable subsets $E \subset \Omega$, the following properties hold:

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(1) $\rho(f) = 0 \Leftrightarrow f = 0$ a.e. in Ω
(2) $\rho(\lambda f) = \lambda\rho(f)$
(3) $\rho(f+g) \le \rho(f) + \rho(g)$
(4) $0 \le g \le f$ a.e. in $\Omega \Rightarrow \rho(g) \le \rho(f)$
(5) $0 \le f_n \uparrow f$ a.e. in $\Omega \Rightarrow \rho(f_n) \uparrow \rho(f)$
(6) $E \subset \Omega \Rightarrow \rho(\chi_E) < +\infty$
(7) $E \subset \Omega \Rightarrow \int_E f dx \le C_E \rho(f)$

for some constant C_E, $0 < C_E < \infty$, depending on E and ρ, but independent of f.

Grand Lebesgue spaces belong to a special category of Banach Function Spaces: they are rearrangement-invariant, namely, setting

$$\mu_f(\lambda) = |\{x \in \Omega : |f(x)| > \lambda\}| \quad \forall \lambda \ge 0 \tag{1.1}$$

it is $\rho(f) = \rho(g)$ whenever $\mu_f = \mu_g$.

A generalization of the grand Lebesgue spaces are the spaces $L^{p),\theta}$, $\theta \ge 0$, defined by (see e.g. [14])

$$\|f\|_{L^{p),\theta}(\Omega)} = \sup_{0<\varepsilon<p-1} \left(\varepsilon^\theta \frac{1}{|\Omega|} \int_\Omega |f|^{p-\varepsilon} dx \right)^{\frac{1}{p-\varepsilon}}.$$

When $\theta = 0$ the spaces $L^{p),0}(\Omega)$ reduce to Lebesgue spaces $L^p(\Omega)$ and when $\theta = 1$ the spaces $L^{p),1}(\Omega)$ reduce to grand Lebesgue spaces $L^{p)}(\Omega)$.

A useful property of the norm, used in [15,16] is the fact that the supremum over $(0, p-1)$ in the norm of $L^{p)}(\Omega)$ can be computed also in any smaller interval $(0, \varepsilon_0)$: the result is an equivalent expression of the norm (i.e. each expression can be majorized by the other, multiplied by a constant not depending on f). Obviously, the constants involved in the equivalence will depend on p and ε_0. This phenomenon has been clarified also in a more general context in [17].

We recall also the continuous embeddings, easy consequence from the definition,

$$L^p(\Omega) \subset L^{p),\theta}(\Omega) \subset L^{p-\epsilon}(\Omega), \quad 0 < \epsilon \le p-1 \theta > 0.$$

2. The main results

Let $\delta : (0, p-1) \to [0, +\infty[$ be a measurable function, and for all $f \in \mathcal{M}_0^+$ set

$$\rho_{p),\delta}(f) = \operatorname*{ess\,sup}_{0<\varepsilon<p-1} \delta(\varepsilon)^{\frac{1}{p-\varepsilon}} \left(\fint_\Omega f^{p-\varepsilon} dx \right)^{\frac{1}{p-\varepsilon}}, \tag{2.1}$$

where $\fint_\Omega$ stands for $\frac{1}{|\Omega|}\int_\Omega$. For $1 \le r < \infty$, we will also write $\|f\|_r$ to denote the *normalized* norm of f in $L^r(\Omega)$:

$$\|f\|_r = \left(\fint_\Omega f^r dx \right)^{\frac{1}{r}}.$$

By convention, we establish that the right hand side of (2.1) is ∞ if for some $0 < \varepsilon < p-1$ the function $f \notin L^{p-\varepsilon}(\Omega)$: this position gives always a meaning to the ess sup, also when the indeterminate form $0 \cdot \infty$ appears. The case $\delta(\varepsilon) = \varepsilon^\theta$, $\theta > 0$, gives back the norm of the $L^{p),\theta}(\Omega)$ spaces.

The first goal of this paper is to find a necessary and sufficient condition on δ such that $\rho_{p),\delta}$ is equivalent to a *Banach function norm*, i.e. equivalent to a functional satisfying all the properties (1)–(7) listed in the previous section.

It is clear that the first way to prove that $\rho_{p),\delta}$ is equivalent to a Banach function norm is to try to reproduce the analogous proof, valid for grand L^p spaces. This latter proof is an easy consequence of the classical properties of the norm of Lebesgue spaces, and it seems, for this reason, almost absent in literature. The problem when considering the functional $\rho_{p),\delta}$ is that δ is defined almost everywhere, and the expression $\delta(\varepsilon)$ does not have the meaning of value attained in ε. Moreover, the estimate of $\rho_{p),\delta}$ looks much less evident when, for instance, δ attains the value zero infinite times in a neighborhood of zero.

Besides solving completely the problems above, we will show in particular that for any measurable bounded δ, $\rho_{p),\delta}$ is equivalent to a Banach function norm, and that the same resulting space can be obtained by using a new function $\bar{\delta}$, defined everywhere, whose expression is explicitly shown. After this step, also in the case of bounded measurable functions, the proof of being equivalent to a Banach function norm can be considered equally trivial as in the case of grand Lebesgue spaces.

Going back to our first goal, an immediate necessary condition is suggested by property (6), when $E = \Omega$: since $\rho_{p),\delta}(\chi_\Omega)$ must be finite, it must be $\delta \in L^\infty(0, p-1)$. The Theorem we will prove is that this condition is actually also sufficient.

Theorem 2.1. *Let* $1 < p < \infty$ *and let* $\delta : (0, p-1) \to [0, +\infty[$ *be a measurable function, not identically zero. The mapping* $\rho_{p),\delta}$ *is equivalent to a Banach function norm if and only if* $\delta \in L^\infty(0, p-1)$.