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Positive stationary solutions and threshold results for the non-homogeneous semilinear parabolic equation with Robin boundary conditions[☆]

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ABSTRACT

Let Ω be a smooth bounded domain in R^n. Considering the following Robin problem for a semilinear parabolic equation

$$\begin{cases} u_t - \Delta u = u^p + f(x), & (x,t) \in \Omega \times (0,T), \\ \dfrac{\partial u}{\partial \nu} + \beta u = 0, & (x,t) \in \partial\Omega \times [0,T), \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{0.1}$$

we show that for any function $f(x)$ satisfying $(\mathcal{F})$ which will be given in the introduction, there exists a positive number $\beta_f^\star$ such that problem (0.1) has no stationary solution if $\beta \in (0, \beta_f^\star)$, and has at least two stationary solutions when $\beta > \beta_f^\star$. Moreover, among all stationary solutions of problem (0.1) there is a minimal one. We prove further that the minimal stationary solution of problem (0.1) is stable, whereas, any other stationary solutions of problem (0.1) are initial datum thresholds for the existence and non-existence of a global solution to problem (0.1).

1. Introduction

Let $n \geq 2$, and Ω be a bounded domain in R^n with $C^{2,\alpha}$ boundary $\partial\Omega$. We consider the following Robin problem

$$\begin{cases} u_t - \Delta u = u^p + f(x), & (x,t) \in \Omega \times (0,T), \\ \dfrac{\partial u}{\partial \nu} + \beta u = 0, & (x,t) \in \partial\Omega \times [0,T), \\ u(x,0) = u_0(x) \geq 0, & x \in \Omega, \end{cases} \tag{1.1}$$

where ν indicates the unit outward normal vector on $\partial\Omega$, $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator, $1 < p < \frac{n+2}{n-2}$, $\beta \geq 0$ is a parameter and $f(x) \geq 0, f(x) \not\equiv 0$ is a given function in $C^1(\overline{\Omega})$. Problem (1.1) has different names depending on

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the different values of the parameter β. It is called Dirichlet if $\beta = +\infty$, is called Neumann in the case of $\beta = 0$ and is called Robin provided that $0 < \beta < +\infty$. It is also worth pointing out that problem (1.1) occurs in various branches of mathematical physics and biological models.

Our purpose are two-folded: one is to get the existence and non-existence of positive stationary solutions of problem (1.1); the other is to investigate the stability and instability of stationary solutions of problem (1.1), in particular, to prove a threshold result for problem (1.1).

Firstly, we study the existence and non-existence of positive stationary solutions to problem (1.1). That is, we discuss the existence and non-existence of positive solutions to the following elliptic Robin problem

$$\begin{cases} -\Delta u = u^p + f(x), & x \in \Omega, \\ u > 0, & \in \Omega, \\ \dfrac{\partial u}{\partial \nu} + \beta u = 0 & x \in \partial\Omega. \end{cases} \tag{1.2}$$

It is trivial to observe that problem (1.2) has no solution when $\beta = 0$. In the case $\beta = +\infty$, there are many papers on the existence of positive solutions and multiplicity results for problem (1.2). See for example [1–7] and the references therein. In 1992, Tarantello (see [6]) considered the following Dirichlet problem

$$\begin{cases} -\Delta u = |u|^{p-1}u + f(x), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.3}$$

and proved that problem (1.3) admits at least two solutions $u_0(x), u_1(x) \in H_0^1(\Omega)$ provided that $1 < p \le \frac{n+2}{n-2}$, $f(x) \not\equiv 0$ and

$$\int_\Omega fu dx \le C_n \|\nabla u\|_{L^2(\Omega)}^{(n+2)/2}, \quad \forall u \in H_0^1(\Omega),$$

where $C_n = \frac{4}{n-2}\left(\frac{n-2}{n+2}\right)^{(n+2)/4}$. Moreover, $u_0(x),\ u_1(x) > 0$ when $f(x) \ge 0$ and $f(x) \not\equiv 0$.

It is well-known that problem (1.3) has a unique positive solution when $f(x) \equiv 0$ and Ω is a ball. Hence, roughly speaking, the appearance of the non-homogeneous term $f(x)$ in problem (1.3) increases the number of solutions.

Comparing with the Dirichlet problem, there are not many results about the so called Robin problem. However, Gu and Liu [8] studied the following Robin problem

$$\begin{cases} -\Delta u = u^p, & x \in \Omega, \\ u > 0, & x \in \Omega, \\ \dfrac{\partial u}{\partial \nu} + \beta u = 0, & x \in \partial\Omega. \end{cases} \tag{1.4}$$

They proved that the solution set of problem (1.4) is compact in $L^\infty(\Omega)$. By the compactness of the solution set and the fixed point theorem, they also got that problem (1.4) has at least one solution for any $0 < \beta < +\infty$ and $1 < p < \frac{n+2}{n-2}$. It is also worthy of mention that some uniqueness and multiplicity results for problem (1.4) with $p \in \left(1, \frac{n+2}{n-2}\right)$ can be found in [9,10].

In this paper, we find that, unlike the Dirichlet problem, the presence of non-homogeneous term $f(x)$ in Robin problem can increase, as well as, decrease the number of solutions subject to the range of the parameter β. For exact statements, see Theorems 1.1 and 1.2 below.

Secondly, we turn our attention to problem (1.1) itself. When $f(x) \equiv 0$, problem (1.1) is reduced to the following model problem

$$\begin{cases} u_t - \Delta u = u^p, & (x,t) \in \Omega \times (0,T), \\ \dfrac{\partial u}{\partial \nu} + \beta u = 0, & (x,t) \in \partial\Omega \times [0,T), \\ u(x,0) = u_0(x) \ge 0, & x \in \Omega. \end{cases} \tag{1.5}$$

This model problem has been studied by many authors (see for example [11–17]), and some threshold results were obtained. In [13], Liu and Zhao got a threshold result for problem (1.5) described by the energy of initial data $u_0(x)$. They proved that the energy of the least energy stationary solution of problem (1.5) is a threshold energy for the energy of initial data $u_0(x)$ to ensure the existence and non-existence of global solution to problem (1.5). Nevertheless, there are few results on higher energy case. In [12], A.A. Lacey showed that any positive stationary solution of problem (1.5) is an initial datum threshold for the existence and non-existence of its global solution. By similar arguments to that used by A.A. Lacey, it is easy to verify that the same conclusion holds for $\beta = +\infty$ in problem (1.5), see [11] for detailed account.

Our threshold result about problem (1.1) differs from that of [12] due to the particularity of the structure of solution set for problem (1.2). In fact, we observe that there is a positive number β_f^* such that problem (1.2) has a unique minimal solution for $\beta \ge \beta_f^\star$, which separates from any other positive solutions. Moreover, any two distinct solutions of problem (1.2) which differ from the minimal one must intersect somewhere. Based on this observation, we prove that the minimal solution of problem (1.2) is stable and any other positive solution of problem (1.2) is an initial datum threshold for the existence and non-existence of global solutions to problem (1.1) when $\beta \ge \beta_f^\star$. What is more, our proof is based on a delicate combination