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Self-adjoint operators on real Banach spaces

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ABSTRACT

The aim of this paper is to discuss a functional equation

$$\rho'_+(f(x), y) = \rho'_+(x, f(y))$$

for all $x, y \in X$. We show that, if a mapping $f: X \to X$ satisfies this functional equation, then f must be a linear continuous operator and we solve this equation in the case when $X = C(M)$. Moreover, we give a new characterization of inner product spaces.

1. Introduction

Let $(X, \|\cdot\|)$ be a real normed space. We define two mappings $\rho'_+, \rho'_-: X \times X \to \mathbb{R}$:

$$\rho'_\pm(x, y) := \lim_{t\to 0^\pm} \frac{\|x + ty\|^2 - \|x\|^2}{2t} = \|x\| \cdot \lim_{t\to 0^\pm} \frac{\|x + ty\| - \|x\|}{t}.$$

These mappings are called *norm derivatives*. Now, we recall their useful properties (proofs can be found in [1,2]):

(nd1) $\forall_{x,y\in X}\ \forall_{\alpha\in\mathbb{R}}\ \rho'_\pm(x, \alpha x + y) = \alpha\|x\|^2 + \rho'_\pm(x, y)$;
(nd2) $\forall_{x,y\in X}\ \forall_{\alpha\geqslant 0}\ \rho'_\pm(\alpha x, y) = \alpha\rho'_\pm(x, y) = \rho'_\pm(x, \alpha y)$;
(nd3) $\forall_{x,y\in X}\ \forall_{\alpha<0}\ \rho'_\pm(\alpha x, y) = \alpha\rho'_\mp(x, y) = \rho'_\mp(x, \alpha y)$;
(nd4) $\forall_{x\in X}\ \rho'_\pm(x, x) = \|x\|^2$;
(nd5) $\forall_{x,y\in X}\ |\rho'_\pm(x, y)| \leqslant \|x\| \cdot \|y\|$;
(nd6) $\forall_{x,y\in X}\ \rho'_-(x, y) \leqslant \rho'_+(x, y)$;
(nd7) $\forall_{x,y,z\in X}\ \rho'_+(x, y + z) \leqslant \rho'_+(x, y) + \rho'_+(x, z)$;
(nd8) $\forall_{x,y,z\in X}\ \rho'_-(x, y + z) \geqslant \rho'_-(x, y) + \rho'_-(x, z)$;

Moreover, mappings ρ'_+, ρ'_- are continuous with respect to the second variable, but not necessarily with respect to the first one. Note, that if $(X, \langle\cdot|\cdot\rangle)$ is an inner product space, then $\langle y|x\rangle = \rho'_+(x, y) = \rho'_-(x, y)$ for arbitrary $x, y \in X$.

In a real normed space, one can define various orthogonality relations. In the paper, we will consider *the Birkhoff–James orthogonality*:

$$x \perp_B y :\Leftrightarrow \forall_{\lambda\in\mathbb{R}} \|x\| \leqslant \|x + \lambda y\|.$$

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In a real normed space X we have for arbitrary $x, y \in X$ (see [2]):

$$\rho'_-(x, y) \leqslant 0 \leqslant \rho'_+(x, y) \Leftrightarrow x \perp_B y. \tag{1.1}$$

In a similar way as in the inner product space, we introduce ρ_+-*orthogonality* and ρ_--*orthogonality* (see [3,2]):

$$x \perp_{\rho_+} y :\Leftrightarrow \rho'_+(x, y) = 0, \qquad x \perp_{\rho_+} y :\Leftrightarrow \rho'_-(x, y) = 0.$$

In an inner product space we have $\perp = \perp_{\rho_+} = \perp_{\rho_+} = \perp_B$. In an arbitrary normed space we have $\perp_{\rho_+}, \perp_{\rho_+} \subset \perp_B$.

2. Open problem

Let $(H, \langle\cdot|\cdot\rangle)$ be a Hilbert space. It is easy to check that, if $f : H \to H$ satisfies

$$\forall_{x,y\in X} \quad \langle f(x)|y\rangle = \langle x|f(y)\rangle,$$

then f is linear and continuous. In this paper, we will give a natural generalization of such a functional equation in the case of real normed spaces. The following problem was formulated in [1, p. 177].

Problem 2.1. Find all functions $f : X \to X$ such that

$$\rho'_+(f(x), y) = \rho'_+(x, f(y)),$$

for all x, y in X.

Further on, we will give a characterization of inner product spaces. Namely, we will answer the question posed in [1, p. 177].

Problem 2.2. If this functional equation holds for all linear transformations f whose matrices are symmetric, does $\|\cdot\|$ derive from an inner product?

3. Preliminaries

3.1. On some properties of the norm

A normed space $(X, \|\cdot\|)$ is said to be *smooth at the point* $x_o \in X \setminus \{0\}$, if there is a unique $x^* \in X^*$ such that $x^*(x_o) = \|x_o\|$ and $\|x^*\| = 1$. Now we will give a characterization of smoothness at a point in terms of the norm derivatives (see [2,1]).

Theorem 3.1. *Let $(X, \|\cdot\|)$ be a real normed space and let $x_o \in X \setminus \{0\}$. Then, the following statements are equivalent:*

(1) *X is smooth at the point x_o;*
(2) *the norm is Gâteaux differentiable at x_o;*
(3) *$\forall_{y\in X}\ \rho'_-(x_o, y) = \rho'_+(x_o, y)$;*
(4) *the functional $\rho'_+(x_o, \cdot)$ is linear;*
(5) *the functional $\rho'_-(x_o, \cdot)$ is linear;*
(6) *X is smooth at the point $-x_o$.*

Now, we consider a set

$$D_{sm}(X) := \{x \in X : X \text{ is smooth at } x\} \cup \{0\}.$$

Throughout this paper, we will often assume that the set $D_{sm}(X)$ is dense. We will show that the separable Banach space has this property.

The following theorem of Mazur (see [4] or [5, p. 12]) will be useful for further considerations.

Theorem 3.2. *Let $(X, \|\cdot\|)$ be a separable Banach space and $f : D \to \mathbb{R}$ be a continuous convex function on an open convex subset D of X. Then there exists a dense G_δ set $G \subset D$ such that f is Gâteaux differentiable on G.*

Clearly the norm $\|\cdot\| : X \to \mathbb{R}$ is convex. Therefore, applying the Theorem of Mazur and Theorem 3.1, we get:

Theorem 3.3. *Let $(X, \|\cdot\|)$ be a separable real Banach space. Then $D_{sm}(X)$ is dense.*

In this paper, a set $E \subset X^*$ is called *a total set*, if

$$\forall_{x\in X\setminus\{0\}} \exists_{\varphi\in E} : \varphi(x) \neq 0,$$

or equivalently, if

$$\forall_{x\in X} \quad \left[\left(\forall_{\varphi\in E} \varphi(x) = 0\right) \Rightarrow x = 0\right].$$

Let us denote the set $\left\{\rho'_+(x, \cdot) \in X^* : x \in D_{sm}(X)\right\}$ by X^*_{sm}. The following lemma will be useful in the proof of the main result.