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Convergence to equilibrium in degenerate parabolic equations with delay

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ABSTRACT

In [11], Busenberg & Huang (1996) showed that small positive equilibria can undergo supercritical Hopf bifurcation in a delay-logistic reaction–diffusion equation with Dirichlet boundary conditions. Consequently, stable spatially inhomogeneous time-periodic solutions exist. Previously in [12] Badii, Diaz & Tesei (1987) considered a similar logistic-type delay-diffusion equation, but differing in two important respects: firstly by the inclusion of nonlinear degenerate diffusion of so-called porous medium type, and secondly by the inclusion of an additional 'dominating instantaneous negative feedback' (where terms local in time majorize the delay terms, in some sense). Sufficient conditions were given ensuring convergence of non-negative solutions to a unique positive equilibrium.

A natural question to ask, and one which motivated the present work, is: can one still ensure convergence to equilibrium in delay-logistic diffusion equations in the presence of nonlinear degenerate diffusion, but in the absence of dominating instantaneous negative feedback? The present paper considers this question and provides sufficient conditions to answer in the affirmative. In fact the results are much stronger, establishing global convergence for a much wider class of problems which generalize the porous medium diffusion and delay-logistic terms to larger classes of nonlinearities. Furthermore the results obtained are independent of the size of the delay.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$ with boundary of class $C^{2+\alpha}$ for some $\alpha \in (0,1)$. Define $Q_T = \Omega \times (0,T]$, $S_T = \partial\Omega \times (0,T]$ and $\Gamma_r = \overline{\Omega} \times [-r,0]$. We consider the following nonlinear degenerate diffusion equation with delay

$$\text{(D)} \quad \begin{cases} \partial_t u = \Delta\phi(u) + f(u(x,t))h(u(x,t-r)) & \text{in } Q_T, \\ u = 0 & \text{in } S_T, \\ u = \eta_s \geq 0 & \text{in } \Gamma_r, \end{cases}$$

where $r > 0$ is the delay and $\eta_s(x) := \eta(x,s)$ the initial data. As usual Δ denotes the Laplacian operator and $\partial_t u$ denotes the partial time derivative $\partial u/\partial t$. Throughout we will write Q instead of Q_∞ and we will sometimes abuse notation slightly by writing $u(t)$ instead of $u(\cdot,t)$, for a function $u(x,t)$.

The associated time-independent stationary problem for (D) is given by

$$\text{(DS)} \quad \begin{cases} \Delta\phi(u) + f(u)h(u) = 0 & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega. \end{cases}$$

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This paper is concerned with the large-time behaviour of non-negative solutions of (D) and their convergence to solutions of (DS) as $t \to \infty$. Its novelty lies in the combination of three distinct types of nonlinearity: degenerate diffusion $\phi(u)$, local source term $f(u)$ and time delay $h(u(x, t-r))$. Such problems are often considered as models in population dynamics where $\phi(u)$ represents movement of individuals to avoid over-crowding, $f(u)$ an intrinsic growth rate and $h(u(x, t-r))$ a delayed response due to gestation periods, resource conversion, incubation periods, etc.

There is a large literature relating to problem (D) when delays are absent ($r = 0$), see [1–3] for an overview and extensive bibliographies. We will make use of several key results from this literature on degenerate parabolic equations, using it mainly to provide suitable comparison solutions for the solutions of (D). Several authors have considered the non-delay degenerate parabolic case in the presence of periodic forcing terms and established existence and attractivity properties of periodic solutions [4,5]. There are also many works dealing with the case of linear diffusion and nonlinear delay terms ($r > 0$), see [6–8] for an overview and references. More recently researchers have considered problems incorporating degenerate diffusion, delay and periodic forcing [9,10].

In [11] the authors considered the following linear diffusion case $\phi(u) = u$ with logistic delay $h(u) = 1 - u$ and local source term $f(u) = ku$ ($k > 0$):

$$\partial_t u = u_{xx} + ku(x, t)(1 - u(x, t-r)), \quad x \in (0, \pi),\ t > 0, \tag{1}$$

$$u = 0, \qquad x = 0, \pi, \qquad t > 0. \tag{2}$$

It is well known that (1)–(2) possesses a unique positive equilibrium U_k for all $k > 1$, and only the trivial equilibrium $U = 0$ when $k < 1$, with $\|U_k\|_\infty \to 0$ as $k \to 1^+$. In the case of no delay ($r = 0$) it is also well known that U_k attracts all non-negative non-trivial solutions for $k > 1$; when $k < 1$ the trivial equilibrium $U = 0$ attracts such solutions. In [11] the authors fixed k slightly greater than, sufficiently close to, 1 and showed that U_k undergoes Hopf bifurcation as r increases through an infinite sequence of positive values $0 < r_0(k) < r_1(k) < r_2(k), \ldots$. In particular they showed that the first bifurcation at $r_0(k)$ is supercritical, giving rise to stable, spatially inhomogeneous, time-periodic solutions of (1)–(2). Consequently, there exist values of k and of the delay parameter r for which the (small) positive equilibrium U_k is not locally attractive. The present work was motivated in part by asking whether this kind of 'delay-induced instability' can occur when linear diffusion is replaced by nonlinear degenerate diffusion, such as $\Delta(u^m)$ for $m > 1$ (the so-called porous medium slow diffusion operator).

No comprehensive literature exists for degenerate parabolic equations including delay terms (and without periodic forcing). To the best of the author's knowledge the only paper in a similar spirit to the present one is [12]. There the authors considered the equation

$$\partial_t u = \Delta u^m + u\left(a(x) - b(x)u - \int_{-\infty}^{t} u(x, s)K(x, t-s)\, ds\right), \tag{3}$$

where b and K are non-negative functions and the positivity set of a in Ω is non-empty. Crucially, and in contrast to the present paper where $b \equiv 0$, it was assumed in [12] that $b > 0$ on $\overline{\Omega}$. Furthermore, in order to guarantee convergence to a unique positive equilibrium, b was assumed [12, Theorem 2.5] to satisfy the stronger condition

$$b(x) \geq \int_0^\infty K(x, s) ds, \quad \forall x \in \Omega. \tag{4}$$

Assumptions such (4) are sometimes referred to as 'diagonally dominant' or having 'negative instantaneous feedback' in the delay-differential equation literature and in the theory of competitive population dynamics. Mathematically this property is often used to overcome the absence of a comparison principle in situations where the delay term has a negative response effect, corresponding to $h' < 0$ in our context. See [13–15,6].

Such terminology refers to the assumption that the local, instantaneous term bu dominates the non-local, delayed term $K \star u$ (the convolution term in (3)). The work of [12] provides a second motivation for the present paper, namely to obtain sufficient conditions for global convergence of non-negative solutions in the absence of negative instantaneous feedback (i.e. with $b = 0$.)

The remainder of the paper is structured as follows. In Section 2 we define the solution concepts for the problems encountered and establish preliminary existence-uniqueness results. In Section 3 we summarize and extend some known results from the literature concerning sign-indefinite degenerate parabolic equations. Section 4 contains the main results of the paper. Sufficient conditions will be given which ensure global convergence of non-negative solutions of (D) to a positive equilibrium, see Theorem 4.1. The class of problems for which the results are applicable include the logistic-type reaction term described above as a special case. The final section, Section 5 contains some examples and discussions.

2. Global existence and uniqueness for the delay problem

Let $\mathbb{R}^+ = [0, \infty)$. We begin with the following assumptions:

(A1) $\phi \in C^1(\mathbb{R}^+)$, $\phi(0) = \phi'(0) = 0$, $\phi' > 0$ on $(0, \infty)$, ϕ^{-1} exists and $\phi^{-1} \in C^\alpha(\mathbb{R}^+)$, there exist $\gamma, \delta > 0$ such that ϕ is convex on $(0, \delta)$ and $u\phi'(u) < \gamma\phi(u)$ on $(0, \infty)$.