Nonlinear Analysis 81 (2013) 247–263

Contents lists available at SciVerse ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Regularity of traveling periodic stratified water waves with vorticity

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ARTICLE INFO

Article history:
Received 2 November 2012
Accepted 5 November 2012
Communicated by Enzo Mitidieri

MSC:
35B65
35Q35
76B03
76B15

Keywords:
Analyticity
Gevrey regularity
Wave profile
Stratified water wave
Vorticity

ABSTRACT

We prove real analyticity of all the streamlines, including the free surface, of a steady stratified flow of water over a flat bed in the absence of stagnation points, with a Hölder continuous Bernoulli function and a Hölder continuously differentiable density function. Furthermore, we show that if the Bernoulli function and the density function possess some Gevrey regularity of index s, then the stream function admits the same Gevrey regularity throughout the fluid domain; in particular if the Gevrey index s equals to 1, then we obtain analyticity of the stream function. The regularity results hold for three distinct physical regimes: capillary, capillary-gravity, and gravity water waves.

1. Introduction

We study regularity of the streamlines, including the surface profile, and regularity of the stream function, of stratified flows. Stratified flows are heterogeneous flows where the density varies as a function of the streamlines [1–3]. Stratification is a physically significant phenomenon for certain flows, where the fluid density may be caused to fluctuate by numerous factors such as the interplay between gravity and the salinity of the water. Recently, in [4,5], the author developed an existence theory for two-dimensional stratified steady and periodic gravity waves, with or without surface tension. Using local and global bifurcation techniques, it was shown that, for stratified flows without stagnation points, there exist both small and large amplitude traveling periodic waves. Even more recently, some results ensuring the local existence of stratified waves, allowing for both surface tension and stagnation points, were proven by the authors in [6], extending the work of [7]. However, in the analysis of this paper, we will invoke the assumption of there being no stagnation points.

For homogeneous water waves, that is, waves with constant density, a series of works analyze the a priori regularity of the streamlines, and particularly the free-surface. For instance in the irrotational setting Lewy [8] showed that irrotational waves without stagnation points have real analytic profiles. Recent developments, proving the regularity of streamlines for rotational flows, were initiated by Constantin and Escher in [9] in the setting of homogeneous gravity waves over a flat bed. Under the assumption that the vorticity function is Hölder continuously differentiable, it was shown in [9] that, each streamline, except the free surface, is real analytic; if further the vorticity function is real analytic, then the free surface itself is also analytic. The arguments in [9] base on translational invariance property of the resulting elliptic operator in the direction of wave propagation, and the celebrated result due to Kinderlehrer et al. [10] on regularity for elliptic free boundary problems. Later on, similar strong a priori regularity was established for a wide variety of homogeneous regimes,

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0362-546X/$ – see front matter
doi:10.1016/j.na.2012.11.009

see for instance [11–13] for periodic gravity waves with surface tension, [14] for deep-water waves, [15] for flows with merely bounded vorticity, [16,17] for solitary-water waves, and the survey article [18]. In all the aforementioned works the analyticity of free surface is established under the extra assumption that the vorticity function is analytic. Quite recently, a similar result was established in [19] without the analyticity assumption on the vorticity function. Precisely, the authors in [19] showed that if the vorticity function is only Höder continuous, all the streamlines, including the free surface, of the steady homogeneous flow over a flat bed in the absence of stagnation points, are real analytic. The conclusions in [19] were achieved by using some a priori Schauder estimates and giving successively a quantitative bound for each derivative of the streamlines in the Höder norm. Moreover, studied also in [19] was the case when the vorticity possesses more regularity property rather than Höder continuity, namely Gevrey regularity of index s. Gevrey class is an intermediate space between the spaces of smooth functions and analytic functions, and the Gevrey class function of index 1 is just the real-analytic function; see Section 2.2 below for precise definition of Gevrey class. It was shown in [19] that if the vorticity is Gevrey regular, the stream function admits the same Gevrey regularity in the fluid domain, up to the free surface.

For heterogeneous, or stratified, water waves, some regularity results of steady periodic waves without stagnation points, have been studied recently in [20], in three distinct physical regimes, namely: capillary, capillary-gravity, and gravity water waves. There the authors proved, for all three types of waves, that, when the Bernoulli function is Höder continuous and the variable density function has a first derivative which is Höder continuous, then the free-surface profile is the graph of a smooth function. Furthermore, they showed that the streamlines are analytic a priori for capillary stratified waves, whereas for gravity and capillary-gravity stratified waves the streamlines are smooth in general, and analytic in an unstable regime; moreover, if the Bernoulli function and the streamline density function are both real analytic functions then all of the streamlines, including the wave profile, are real analytic for all gravity, capillary, and capillary-gravity stratified waves.

In the present work we follow the arguments in [19] for homogeneous water waves to study the regularity of the streamlines, including the surface profile, for stratified flows. We show, for the above three types of waves, that: (1) if the Bernoulli function is Höder continuous and the density function is Höder continuously differentiable, then all the streamlines, including the wave profile, are real analytic; see Theorem 2.2; (2) if both the Bernoulli function and the density function are in Gevrey (analytic) class, then the stream function admits the same Gevrey (analytic) regularity in the fluid domain, up to the free surface; see Theorem 2.3 stated in Section 2.2.

The paper is organized as follows. In Section 2 we formulate the rotational capillary-gravity stratified water-wave problem as the free boundary problem for the stream function and its equivalent reformulation in a fixed rectangular domain, and state our main regularity results. Notations and some useful inequalities are listed. Section 3 is devoted to the proof of analyticity of streamlines including the free surface. In Section 4 we study the Gevrey (analytic) regularity of stream function. In Section 5, we consider the traveling gravity water waves, and obtain similar regularity results for streamlines and stream function.

2. Preliminaries and the main results

2.1. The governing equations for stratified water waves

Consider a steady two-dimensional flow of an incompressible inviscid fluid with variable density and a steady wave on the free surface of the flow. By steady, we mean that the flow and the surface wave move at a constant speed from left to right without changing their configuration; that is, the velocity field of flow and the surface wave exhibit a (t, X)-dependence in the form of $X - ct$, where X is the horizontal space variable, $c > 0$ is the speed of the wave and t denotes the time. Setting $x = X - ct$, we eliminate the time dependence of fluid flow and pass to the frame of reference moving with the wave. Assume that the flow is over a flat bed $y = -d$ with $0 < d < \infty$, the free surface is given by $y = \eta(x)$ which is oscillating around the line $y = 0$, and the liquid occupy the stationary domain

$$\Omega = \{(x, y) \in \mathbb{R}^2 : -d < y < \eta(x)\}.$$

Also, the flow is assumed to be driven by capillarity (that is, surface tension) on the surface and gravity acting on the body of the fluid.

Let $u = u(x, y)$ and $v = v(x, y)$ denote the horizontal and vertical velocities, respectively, and let $\tilde{\rho} = \tilde{\rho}(x, y) > 0$ be the density. Define the *(relative) pseudo-stream function* $\psi(x, y)$ by

$$\psi_y = \sqrt{\tilde{\rho}}(u - c), \qquad \psi_x = -\sqrt{\tilde{\rho}}v. \tag{1}$$

The level sets $\{(x, y) : \psi(x, y) = \text{constant}\}$ are called *streamlines* of the fluid motion. The above relation (1) determines ψ up to a constant. For definiteness we choose $\psi = 0$ on the free boundary, so that $\psi = -p_0$ on $y = -d$, where $p_0 < 0$ is the *(relative) pseudo-volumetric mass flux*:

$$p_0 = \int_{-d}^{\eta(x)} \psi_y(x, y)\, dy.$$

Since ρ is transported, it must be constant on the streamlines and hence, we may think of it as a function of ψ and assume

$$\tilde{\rho}(x, y) = \rho(-\psi(x, y)),$$