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Conformal bounds for the first eigenvalue of the p-Laplacian

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ABSTRACT

Let M be a compact, connected, m-dimensional manifold without boundary and $p > 1$. For $1 < p \leq m$, we prove that the first eigenvalue $\lambda_{1,p}$ of the p-Laplacian is bounded on each conformal class of Riemannian metrics of volume one on M. For $p > m$, we show that any conformal class of Riemannian metrics on M contains metrics of volume one with $\lambda_{1,p}$ arbitrarily large. As a consequence, we obtain that in two dimensions $\lambda_{1,p}$ is uniformly bounded on the space of Riemannian metrics of volume one if $1 < p \leq 2$, respectively unbounded if $p > 2$.

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1. Introduction

Let M be a compact m-dimensional manifold. All through this paper we will assume that M is connected and without boundary. The p-Laplacian ($p > 1$) associated to a Riemannian metric g on M is given by

$$\Delta_p u = \delta(|du|^{p-2} du),$$

where $\delta = -\mathrm{div}_g$ is the adjoint of d for the L^2-norm induced by g on the space of differential forms. This operator can be viewed as an extension of the Laplace–Beltrami operator which corresponds to $p = 2$. The real numbers λ for which the nonlinear partial differential equation

$$\Delta_p u = \lambda |u|^{p-2} u$$

has nontrivial solutions are the *eigenvalues* of Δ_p, and the associated solutions are the *eigenfunctions* of Δ_p. Zero is an eigenvalue of Δ_p, the associated eigenfunctions being the constant functions. The set of nonzero eigenvalues is a nonempty, unbounded subset of $(0, \infty)$ [1]. The infimum $\lambda_{1,p}$ of this set is itself a positive eigenvalue, the *first eigenvalue* of Δ_p, and has a Rayleigh type variational characterization [2]:

$$\lambda_{1,p}(M, g) = \inf \left\{ \frac{\int_M |du|^p \; v_g}{\int_M |u|^p \; v_g} \middle| u \in W^{1,p}(M) \setminus \{0\}, \int_M |u|^{p-2} u \, v_g = 0 \right\},$$

where v_g denotes the Riemannian volume element associated to g.

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The first eigenvalue of Δ_p can be viewed as a functional on the space of Riemannian metrics on M:

$$g \mapsto \lambda_{1,p}(M, g).$$

Since $\lambda_{1,p}$ is not invariant under dilatations ($\lambda_{1,p}(M, cg) = c^{-\frac{p}{2}}\lambda_{1,p}(M, g)$), a normalization is needed when studying the uniform boundedness of this functional. It is common to restrict $\lambda_{1,p}$ to the set $\mathcal{M}(M)$ of Riemannian metrics of volume one on M. In the linear case $p = 2$ this problem has been extensively studied in various degrees of generality. The functional $\lambda_{1,2}$ was shown to be uniformly bounded on $\mathcal{M}(M)$ in two dimensions [3–5], and unbounded in three or more dimensions [6–11]. However, $\lambda_{1,2}$ becomes uniformly bounded when restricted to any conformal class of Riemannian metrics in $\mathcal{M}(M)$ [12].

In the general case $p > 1$, the functional $\lambda_{1,p}$ is unbounded on $\mathcal{M}(M)$ in three or more dimensions [13]. In this paper we study the existence of uniform upper bounds for the restriction of $\lambda_{1,p}$ to conformal classes of Riemannian metrics in $\mathcal{M}(M)$:

- for $1 < p \le m$ we extend the results from the linear case and obtain an explicit upper bound for $\lambda_{1,p}$ in terms of p, the dimension m and the Li–Yau n-conformal volume.
- for $p > m$, we consider first the case of the unit sphere S^m and we construct Riemannian metrics in $\mathcal{M}(S^m)$, conformal to the standard metric *can* and with $\lambda_{1,p}$ arbitrarily large. We use then the result on spheres to show that any conformal class of Riemannian metrics on M contains metrics of volume one with $\lambda_{1,p}$ arbitrarily large.

As a consequence, we obtain that in two dimensions, $\lambda_{1,p}$ is uniformly bounded on $\mathcal{M}(M)$ when $1 < p \le 2$, and unbounded when $p > 2$.

2. The case $1 < p \le m$: Li–Yau type upper bounds

Let g be a Riemannian metric on M and denote by $[g] = \{fg | f \in C^\infty(M),\ f > 0\}$ the conformal class of g. Let $G(n) = \{\gamma \in \mathrm{Diff}(S^n) \mid \gamma^* can \in [can]\}$ denote the group of conformal diffeomorphisms of $(S^n,\ can)$.

For n big enough, the Nash–Moser Theorem ensures (via the stereographic projection) that the set $I_n(M, [g]) = \left\{\phi : M \to S^n \mid \phi^* can \in [g]\right\}$ of conformal immersions from (M, g) to $(S^n,\ can)$ is nonempty. The *n-conformal volume* of $[g]$ is defined by [5]:

$$V_n^c(M, [g]) = \inf_{\phi \in I_n(M,[g])} \sup_{\gamma \in G(n)} Vol\left(M, (\gamma \circ \phi)^* can\right),$$

where $Vol\,(M, (\gamma \circ \phi)^* can)$ denotes the volume of M with respect to the induced metric $(\gamma \circ \phi)^* can$. By convention, $V_n^c(M, [g]) = \infty$ if $I_n(M, [g]) = \emptyset$.

Theorem 2.1. *Let M be an m-dimensional compact manifold and $1 < p \le m$. For any metric $g \in \mathcal{M}(M)$ and any $n \in \mathbb{N}$ we have*

$$\lambda_{1,p}(M, g) \le m^{\frac{p}{2}} (n+1)^{\left|\frac{p}{2}-1\right|} V_n^c(M, [g])^{\frac{p}{m}}.$$

Remark 2.2. In the linear case $p = 2$, this result was proved by Li and Yau [5] for surfaces and by El Soufi and Ilias [12] for higher dimensional manifolds.

Remark 2.3. Theorem 2.1 gives an explicit upper bound for $\lambda_{1,p}$, $1 < p \le m$, in the case of some particular manifolds: the sphere S^m, the real projective space $\mathbb{RP}^m$, the complex projective space $C\mathbb{P}^d$, the equilateral torus $\mathbb{T}^2_{eq}$, the generalized Clifford torus $S^r\left(\sqrt{r/r+q}\right) \times S^q\left(\sqrt{q/r+q}\right)$, endowed with their canonical metrics. For these manifolds we have [12]: $V_n^c(M, [can]) = Vol(M, \frac{\lambda_{1,2}}{m} can)$ for $n+1$ greater or equal to the multiplicity of $\lambda_{1,2}$.

Using the relationships between the conformal volume and the genus of a compact surface [5,14] we obtain:

Corollary 2.4. *Suppose $m = 2$ and $1 < p \le 2$. Then for any metric $g \in \mathcal{M}(M)$*

$$\lambda_{1,p}(M, g) \le k_p \left[\frac{genus(M) + 3}{2}\right]^{\frac{p}{2}},$$

where [ ] denotes the integer part, $k_p = 3^{\left|\frac{p}{2}-1\right|}(8\pi)^{\frac{p}{2}}$ if M is orientable and $k_p = 5^{\left|\frac{p}{2}-1\right|}(24\pi)^{\frac{p}{2}}$ if not.

Remark 2.5. In the case $p = 2$ and $M = S^2$, this result is the well known Hersch inequality [3]. For higher genus surfaces, the upper bound of $\lambda_{1,2}$ in terms of the genus is due to Yang and Yau [4] (see also [14]).

In order to prove Theorem 2.1 we need two lemmas:

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