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Large deviations for stochastic differential delay equations

Chi Mo, Jiaowan Luo *

Department of Probability and Statistics, School of Mathematics and Information Sciences, Guangzhou University, Guangzhou, Guangdong 510006, PR China
Key Laboratory of Mathematics and Interdisciplinary Sciences of Guangdong Higher Education Institutes, Guangzhou University, PR China

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ABSTRACT

In this work, a Freidlin–Wentzell type large deviation principle is established for stochastic differential delay equations. The result in Mohammed and Zhang (2006) [6] is improved.

1. Introduction

Large deviations theory has a long history beginning with the fundamental work of Donsker and Varadhan [1–3] and, the general large deviation principle (**LDP**) was first formulated by Varadhan [4] in 1966. Subsequently, their applications to stochastic differential equations (SDEs) were first studied by Freidlin and Wentzell [5] in the finite dimensional setting, that is, where the SDEs are driven by finitely many Brownian motions. Since the original work of Freidlin and Wentzell, the finite dimensional problem has been extensively studied and many of the original assumptions made in [5] have been significantly relaxed (see [6,7]).

It is worth noting that several authors have established the **LDP** for a class of stochastic differential equations (see e.g. [6,8–12]). The proofs of **LDP** have usually relied on first approximating the original problem by time discretization so that **LDP** can be shown for the resulting simpler problems via the contraction principle, and then showing that **LDP** holds in the limit. The discretization method used to establish **LDP** was invented by Freidlin and Wentzell [5].

Dupuis and Ellis [7] have recently combined weak convergence methods with the stochastic control approach developed earlier by Fleming [13], who considered functionals of nondegenerate diffusions that satisfied certain parabolic partial differential equations and applied a representation to study certain large deviation problems, for the large deviations theory. A side benefit of this approach is that one can often prove large deviation properties under weaker conditions than the usual proofs based on discretization and approximation arguments, since the exponential continuity in probability and exponential tightness estimates that are used in usual proofs to justify approximations are often obtained under additional conditions on the model than those needed for well-posedness and compactness.

* Corresponding author. Tel.: +86 20 39366221; fax: +86 20 39366222.
E-mail addresses: mathmo@foxmail.com (C. Mo), mathluo@yahoo.com, jluo@gzhu.edu.cn (J. Luo).

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The weak convergence approach has been used to study the large deviations for homeomorphism flows of non-Lipschitz SDEs by Ren and Zhang [12], for two-dimensional stochastic Navier–Stokes equations by Sritharan and Sundar [14], and stochastic evolution equations with small multiplicative noise by Liu [11]. For more references on this approach we may refer to [7,8,15,16].

Recently, using the approximating approach, Mohammed and Zhang [6] established a Freidlin–Wentzell type **LDP** for the following stochastic differential delay equations (SDDEs):

$$\begin{cases} dX(t) = b(t, X(t), X(t-\tau))dt + \sqrt{\epsilon}\sigma(t, X(t), X(t-\tau))dW(t), & t \in (0, T], \\ X(t) = \phi(t), & t \in [-\tau, 0]. \end{cases}$$

To obtain suitable exponential tail probability estimates for certain stochastic convolutions, the following condition

(A2) the functions $b(\cdot, x, y)$, $\sigma(\cdot, x, y)$ are continuous on $[0, \infty)$, uniformly in $x, y \in \mathbb{R}^d$, i.e.,

$$\lim_{s\to t} \sup_{x,y\in\mathbb{R}^d} |b(s, x, y) - b(t, x, y)| = 0,$$

$$\lim_{s\to t} \sup_{x,y\in\mathbb{R}^d} |\sigma(s, x, y) - \sigma(t, x, y)| = 0,$$

is imposed on b and σ in [6]. An important point is that this condition is not needed for unique solvability of the SDDEs. In the present paper, we move on to study this SDDEs by utilizing the weak convergence approach which does not require any exponential probability estimates. Hence, the assumptions on b and σ in (A2) are no longer needed, i.e, the result in [6] is improved. Our proof is inspired by Liu [11] and Sritharan [14].

The organization of the paper is as follows. Section 2 contains some background material on large deviations. We recall some basic definitions and inequalities and the equivalence between a **LDP** and Laplace principle for a family of probability measures on some Polish space. Section 3 gives a general large deviation result for SDDEs with small additive noise. In Section 4, we establish a Freidlin–Wentzell type **LDP** for SDDEs with small multiplicative noise.

Throughout the paper, the generic constants (with or without indexes), whose values are not important, may be different from line to line. If it is essential, we will write the dependence of the constant on parameters explicitly.

2. Preliminaries

In this section, we present some standard definitions and results from the theory of large deviations.

Throughout this paper, denote by $\tau > 0$ a fixed constant, and $\phi \in C([-\tau, 0]; \mathbb{R})$ a given continuous function on $[-\tau, 0]$. In the following, we will work in the finite time interval $[-\tau, T]$. We shall consider the large deviation principle for the following differential delay equation on $\mathbb{R}$:

$$\begin{cases} dX(t) = b\big(t, X(t), X(t-\tau)\big)dt, & t \in (0, T], \\ X(t) = \phi(t), & t \in [-\tau, 0]. \end{cases} \tag{2.1}$$

Here $b : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a Broel measurable function satisfies the standard linear growth and a Lipschitz condition, that is, there exists a constant L_1 such that the following condition holds for all $x_1, x_2, y_1, y_2 \in \mathbb{R}$ and $t \in [0, T]$,

$$|b(t, x_1, y_1) - b(t, x_2, y_2)| \le L_1 \left(|x_1 - y_1| + |x_2 - y_2|\right).$$

Let $\{\Omega, \mathfrak{F}, \mathbf{P}\}$ be a complete filtered probability space equipped with a complete family of right continuous increasing sub σ-algebras $\{\mathfrak{F}_t, t \in [0, T]\}$ satisfying $\{\mathfrak{F}_t \subset \mathfrak{F}\}$, $\{X^\epsilon\}$ be a family of random variables defined on this space and taking values in some space E.

Definition 2.1 (*Rate Function*)**.** A function $I : E \to [0, +\infty]$ is called a rate function if I is lower semicontinuous. A rate function I is called a good rate function if for each $a < \infty$, the level set $\{f \in E : I(f) \le a\}$ is compact.

Definition 2.2 (*Large Deviation Principle*)**.** Let I be a rate function on E. We say the sequence $\{X^\epsilon\}$ satisfies the large deviation principle with rate function I if the following two conditions hold:

1. *Large deviation upper bound*. For each closed subset F of E,

$$\limsup_{\epsilon\to 0} \epsilon \log \mathbf{P}(X^\epsilon \in F) \le -I(F).$$

2. *Large deviation lower bound*. For each open subset G of E,

$$\liminf_{\epsilon\to 0} \epsilon \log \mathbf{P}(X^\epsilon \in G) \ge -I(G).$$

Definition 2.3 (*Laplace Principle*)**.** Let I be a rate function on E. We say the sequence $\{X^\epsilon\}$ satisfies the Laplace principle with rate function I if for all real-valued bounded continuous functions h defined on E,

$$\lim_{\epsilon\to 0} \epsilon \log \mathbf{E}\left\{\exp\left[-\frac{1}{\epsilon}h(X^\epsilon)\right]\right\} = -\inf_{f\in E}\big\{h(f) + I(f)\big\}.$$