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On multiple solutions of a semilinear Schrödinger equation with periodic potential

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ABSTRACT

This paper is concerned with the semilinear Schrödinger equation

$$(S)\; -\Delta u + V(x)u = f(x,u), \quad u \in H^1(\mathbb{R}^N),$$

where V and f are periodic in the x-variables, f is a superlinear and subcritical nonlinearity, and 0 lies in a spectral gap of $-\Delta u + V$. It is shown that, if f is odd in u then (S) has infinitely many large energy solutions. The proof relies on a generalized variant fountain theorem for strongly indefinite functionals, established in this paper.

1. Introduction

In this paper we consider the semilinear Schrödinger equation

$$\begin{cases} -\Delta u + V(x)u = f(x,u), \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{S}$$

to which the time dependent nonlinear Schrödinger equation is reduced when one is looking for steady state or standing wave solutions. This equation arises (when $N = 3$) for example in nonlinear optics, where it models some phenomena such as laser-induced plasmas and optical channeling of lasers (see [1]). Our assumptions on f and V stated below imply that the solutions of (S) are critical points of a strongly indefinite $\mathcal{C}^1$-functional defined on the Sobolev space $H^1(\mathbb{R}^N)$. We are interested in the multiplicity of solutions of (S) without the following superquadraticity condition due to Ambrosetti and Rabinowitz [2]:

$$\exists \mu > 2 \text{ such that } 0 < \mu F(x,u) \leqslant uf(x,u), \quad \forall u \in \mathbb{R} \setminus \{0\},\ x \in \mathbb{R}^N, \tag{1}$$

where $F(x,u) = \int_0^u f(x,s)ds$. It is well known that condition (1) is mainly used to ensure the boundedness of the Palais–Smale sequences of the energy functional, and without it the problem becomes more complicated. However, there are many functions which are superlinear but do not satisfy (1) (see [3–5] for some examples).

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The spectrum $\sigma(-\Delta + V)$ of the Schrödinger operator $-\Delta + V$ is closely related to the potential V, and the relationship between 0 and $\sigma(-\Delta + V)$ is important for our purpose. So we assume that

(V_1) $V \in \mathcal{C}(\mathbb{R}^N, \mathbb{R})$ is 1-periodic in x_i, $i = 1, \ldots, N$;
(V_2) 0 lies in a gap of the spectrum $\sigma(-\Delta + V)$.

It is well known that, under (V_1), the operator $-\Delta + V$ (on $L^2(\mathbb{R}^N)$) has purely continuous spectrum which is bounded below and consists of closed disjoint intervals. Assumption (V_2) allows a decomposition of $H^1(\mathbb{R}^N)$ into $H^1(\mathbb{R}^N) = Y \oplus Z$ such that the quadratic form

$$u \in H^1(\mathbb{R}^N) \mapsto \int_{\mathbb{R}^N} \left(|\nabla u|^2 + V(x)u^2\right) dx$$

is negative and positive on Y and Z respectively. Both Y and Z are infinite-dimensional, so the operator $-\Delta + V$ is strongly indefinite. For the nonlinearity f we make the following assumptions.

(f_1) The function $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is continuous and 1-periodic with respect to each variable x_j, $j = 1, \ldots, N$.
(f_2) There is a constant $c > 0$ such that $|f(x, u)| \leqslant c(1 + |u|^{p-1})$ for all $x \in \mathbb{R}^N$ and $u \in \mathbb{R}$, where $p > 2$ if $N = 1, 2$ and $2 < p < 2^* := \frac{2N}{N-2}$ if $N \geqslant 3$.
(f_3) $f(x, u) = \circ(u)$ uniformly with respect to x as $|u| \to 0$.
(f_4) $\frac{F(x,u)}{|u|^{\mu}} \to \infty$ as $|u| \to \infty$, uniformly in x, where $\mu > 2$.
(f_5) $u \mapsto \frac{f(x,u)}{|u|}$ is strictly increasing on $\mathbb{R} \setminus \{0\}$.
(f_6) For all $x \in \mathbb{R}^N$ and $u \in \mathbb{R}$, $f(x, -u) = -f(x, u)$.

In this paper we study the existence of infinitely many solutions of (S) under the above assumptions. There is a number of papers dealing with problem (S) (see for instance [1,6–12] and the references therein). The existence of infinitely many solutions of (S), under (V_1), (V_2), (f_1)–(f_6), was first proved by Szulkin and Weth in [10], by reducing the indefinite variational problem to a definite one, and then developing a powerful generalized Nehari manifold method to treat strongly indefinite functionals. However we do not know if these solutions are large energy solutions. In [7], Kryszewski and Szulkin obtained infinitely many solutions under stronger conditions including the Ambrosetti–Rabinowitz condition (1). By using the monotonicity method and a critical point theorem developed by Bartsch and Ding in [13], and without the Ambrosetti–Rabinowitz condition, Zhao et al. [14] proved the existence of infinitely many solutions for the following system

$$\begin{cases} -\Delta u + V(x)u = g(x, v), \\ -\Delta v + V(x)v = h(x, u), \\ u(x) \to 0,\ v(x) \to 0 \quad \text{as } |x| \to \infty \end{cases}$$

provided g and h are odd nonlinearities. In [10,12] the authors also obtained, under (V_1), (V_2), (f_1)–(f_5), the existence of ground state solutions by using different methods. To this date and to the best of our knowledge, the strongest results in that direction are due to Schechter [9].

The main result reads as follows.

Theorem 1. *Assume (V_1), (V_2), (f_1)–(f_6). Then problem (S) has infinitely many large energy solutions.*

In order to carry out the proof, we establish an infinite dimensional version of the variant fountain theorem of W. Zou [3] (see Theorem 2.1), which is adapted to the strongly indefiniteness of the energy functional associated to (S), and also provides bounded Palais–Smale sequences. The main ingredients we use are the degree theory and the τ-topology of Kryszewski and Szulkin ([15], Chapter 6), and the monotonicity method introduced by M. Struwe in [16,17] and developed by L. Jeanjean in [18]. Let X be a separable Hilbert space, we will consider just like Zou [3] the following family of $\mathcal{C}^1$-functionals $\Phi_\lambda : X \to \mathbb{R}$ defined by:

$$\Phi_\lambda(u) := L(u) - \lambda J(u), \quad \lambda \in [1, 2],$$

under the same first two assumptions.

(A_1) Φ_λ maps bounded sets to bounded sets uniformly for $\lambda \in [1, 2]$, and $\Phi_\lambda(-u) = \Phi_\lambda(u)$ for every $(\lambda, u) \in [1, 2] \times X$.
(A_2) $J(u) \geqslant 0$ for every $u \in X$; $L(u) \to \infty$ or $J(u) \to \infty$ as $\|u\| \to \infty$.
(A_3) For every $\lambda \in [1, 2]$, Φ_λ is τ-upper semicontinuous and $\nabla\Phi_\lambda$ is weakly sequentially continuous.

Our following abstract result is a variant of a generalized fountain theorem we presented in a recent paper [19].

Theorem 2 (*Variant Fountain Theorem*)**.** *Under assumptions (A_1)–(A_3), if there are $0 < r_k < \rho_k$ such that $b_k(\lambda) > a_k(\lambda)$ for all $\lambda \in [1, 2]$, then $c_k(\lambda) \geqslant b_k(\lambda)$ for all $\lambda \in [1, 2]$. Moreover, for a.e $\lambda \in [1, 2]$, there exists a sequence $(u_k^n(\lambda))_n \subset X$ such that*

$$\sup_n \|u_k^n(\lambda)\| < \infty, \qquad \Phi_\lambda'(u_k^n(\lambda)) \to 0 \quad \textit{and} \quad \Phi_\lambda(u_k^n(\lambda)) \to c_k(\lambda) \quad \textit{as } n \to \infty.$$

$a_k(\lambda)$, $b_k(\lambda)$ and $c_k(\lambda)$ are defined in Section 2.

The paper is organized as follows. In Section 2 we introduce the Kryszewski–Szulkin degree theory and we prove the generalized variant fountain theorem stated above. In Section 3 we apply it to prove Theorem 1, implying the existence of infinitely many solutions for the problem (S).