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H^1-random attractors for stochastic reaction–diffusion equations with additive noise

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ABSTRACT

In this paper, a new theorem obtained by Li and Guo [Y. Li, B. Guo, Random attractors for quasi-continuous random dynamical systems and applications to stochastic reaction–diffusion equations, J. Differential Equations 245(7) (2008) 1775–1800] is employed to some reaction–diffusion equations with additive noise. We prove the unique existence of $\mathcal{D}$-random attractors in H_0^1 for the corresponding random dynamical system. We also obtain the existence of a unique random equilibrium for the generated random dynamical system under some restrictive assumption on the coefficients.

1. Introduction

The purpose of this paper is to investigate the asymptotic behavior of solutions to a reaction–diffusion equation with additive noise, which reads

$$du - \mu\Delta u dt + (f(x,u) + g(x))dt = \sum_{j=1}^{m} h_j(x) dW_j(t), \tag{1.1}$$

with initial condition $u_0(x)$ and homogeneous Dirichlet boundary condition, i.e.,

$$u(\tau, x) = u_0(x), \qquad u(t,x)|_{\partial\mathcal{O}} = 0, \tag{1.2}$$

where $[\tau, t] \subset \mathbb{R}$, $\mathcal{O} \subset \mathbb{R}^n$, $n \in \mathbf{N}_+$, is an open and bounded set with regular boundary $\partial\mathcal{O}$; the unknown $u(t) = u(t,x)$ is a real-valued random process of $x \in D$; $h_j \in L^\infty(\mathcal{O})$; $W(t) = \{W_1(t), W_2(t), \dots, W_m(t)\}$ are mutually independent two-sided real-valued Wiener processes defined on some complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

To study problem (1.1)–(1.2), we assume that, the nonlinearity f in (1.1) is subject to the following growth and dissipative conditions: for $x \in \mathcal{O}$, $u \in \mathbb{R}$,

$$f(x,u)u \geq C_1|u|^p - \phi_1(x), \quad C_1 \in \mathbb{R}^+, \tag{1.3}$$

$$|f(x,u)| \leq C_2|u|^{p-1} + \phi_2(x), \quad C_2 \in \mathbb{R}^+, \tag{1.4}$$

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$$\frac{\partial f}{\partial u}(x,u) \geq C_3, \quad C_3 \in \mathbb{R}, \tag{1.5}$$

$$\left|\frac{\partial f}{\partial x}(x,u)\right| \leq \phi_3(x), \tag{1.6}$$

where $\phi_1 \in L^1(\mathcal{O}) \cap L^{\frac{p}{2}}(\mathcal{O})$, $\phi_2 \in L^2(\mathcal{O}) \cap L^{p'}(\mathcal{O})$, $\phi_3 \in L^2(\mathcal{O})$, with $\frac{1}{p'} + \frac{1}{p} = 1$ and $p \geq 2$.

It is known that, the long time behavior of solutions of stochastic partial differential equations (SPDE) is usually studied in the framework of the theory of random attractor, which was introduced in [1,2], being an extension to random dynamical systems (RDS) of theory of attractors for deterministic equations found in [3–7] for instance. The existences of random attractor for RDS have been richly developed by many authors for all kinds of SPDE, see [1,2,8–21] and references cited there.

Given a probability space, a random attractor is a compact and invariant random set, that attracts every closed random set in the sense of attraction. Roughly speaking, the existence of random attractors is closely related to the continuity of the corresponding RDS. For a continuous RDS, the general existent result on random attractors depends deeply on the existence of a random compact absorbing set, see [1]. This condition can be achieved in some concrete problems by using the Sobolev compactly embedding theorems. By this technique, the existence of random attractors for problem (1.1)–(1.2) in $L^2(\mathcal{O})$ has been studied in [1], but little is known in stronger norm space $H_0^1(\mathcal{O})$ up to now. It is worth mentioning that, in random case, the existence of random attractors in $L^2(\mathbb{R}^n)$ was proved in [9] by a tail estimate method and in deterministic case, the existence of pullback attractors in $H^1(\mathbb{R}^n)$ and $L^2(\mathbb{R}^n)$ was treated in [22], where they proved the asymptotic compactness of solutions in $H^1(\mathbb{R}^n)$ by estimating the bound of the derivative u_t. This idea is also used in [6,7] to obtain the global attractors for p-Laplacian equations.

The main aim of this article is to investigate the existence of random attractors for the corresponding RDS associated with problem (1.1)–(1.2) in $H_0^1(\mathcal{O})$ space. The interesting features of this topic lie in three aspects: (i) The Eq. (1.1) is stochastic. In this case, since the Wiener processes $W(t)$ are continuous but are not differentiable functions in $\mathbb{R}$, it is impossible to obtain the estimate of u_t for problem (1.1)–(1.2) because of the random noise term. So the technique in [22] cannot be used to copy with the random case to obtain the asymptotic compactness of the corresponding RDS in $H_0^1(\mathcal{O})$. (ii) We can only show that, if the initial datum u_0 belongs to $L^2(\mathcal{O})$, the solution $u(t)$ with the initial condition $u(\tau) = u_0$ is in $L^2(\mathcal{O}) \cap H_0^1(\mathcal{O}) \cap L^p(\mathcal{O})$ and has no higher regularity. Hence the Sobolev compactly embedding theorems cannot be employed in $H_0^1(\mathcal{O})$ and the RDS generated by problem (1.1)–(1.2) are no longer compact in $H_0^1(\mathcal{O})$. (iii) The solutions of problem (1.1)–(1.2) are not continuous in $H_0^1(\mathcal{O})$ with respect to initial number.

In this article, we try to overcome these difficulties by using the notions of quasi-continuity and omega-limits set compactness, which was initiated in [11] in the framework of RDS. This type of compactness is equivalent to the notion of asymptotic compactness in some space, and can be proved by checking the flattening condition, see [23]. The continuity property of the RDS in the phase space demanded in [23] can be weakened to the property of quasi-continuity in [11] when the phase space is a separable Banach space. For many RDS, it is quite easy to check the quasi-continuity and the flattening conditions and it seems quite difficult to check the continuity and asymptotic compactness, especially in the Sobolev space $H_0^m(\mathcal{O})(m \geq 1)$ or $L^q(\mathcal{O})$ $(q > 2)$. In this paper, we prove that the RDS corresponding to problem (1.1)–(1.2) is quasi-continuous and omega-limits set compact in $H_0^1(\mathcal{O})$. It is pointed out that, our method with a relatively simple proof of the existence of random attractors, can be used for a variety of other equations.

Random equilibria, which are a special case of omega-limit sets, are the random analog of fixed points for deterministic systems. They generate stationary stochastic orbits, see [24,25]. We show that, if a restrictive assumption is imposed on the coefficient in (1.5) then we obtain a unique random equilibrium which is globally asymptotically stable. These immediately imply the existence of a random attractor which contains only random equilibrium. It is pointed out that, here the space domain $\mathcal{O}$ may be unbounded provided the Poincaré's inequality holds.

We give the outline of this paper. In Section 2, we present some preliminaries for the theory of RDS. In Section 3, we show the existence and uniqueness of a quasi-continuous RDS on $H_0^1(\mathcal{O})$. In Section 4, we give some estimates for the solution operators in given Hilbert space and then obtain corresponding random attractor. In the last part, we show that, the system possesses a unique random equilibrium under a given condition.

2. Preliminaries and abstract results

The basic notion in RDS is a metric dynamical system (MSD) $\theta \equiv (\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$, which is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a group θ_t, $t \in \mathbb{R}$, of measure preserving transformations of $(\Omega, \mathcal{F}, \mathbb{P})$. An MSD θ is said to be ergodic under $\mathbb{P}$ if for any θ-invariant set $F \in \mathcal{F}$ we have either $\mathbb{P}(F) = 0$ or $\mathbb{P}(F) = 1$, where the θ-invariant set is in the sense $\theta_t F = F$ for $F \in \mathcal{F}$ and all $t \in \mathbb{R}$.

Let X be a separable Banach space with norm $\|\cdot\|_X$ and Borel sigma-algebra $\mathcal{B}(X)$, i.e., the smallest σ-algebra on X which contains all open subsets.

Definition 2.1. (1) An RDS on X over an MSD θ is a family of measurable mappings

$$\varphi : \mathbb{R}^+ \times \Omega \times X \to X, \qquad (t, \omega, x) \mapsto \varphi(t, \omega)x$$