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Limit solutions of the Chern–Simons equation

Augusto C. Ponce [a], Adilson E. Presoto [b,*]

[a] Université catholique de Louvain, Institut de Recherche en Mathématique et Physique, Chemin du cyclotron 2, bte L7.01.02, 1348 Louvain-la-Neuve, Belgium

[b] Universidade Federal de São Carlos, Departamento de Matemática, Rodovia Washington Luís, Km 235, São Carlos, SP, 13565-905, Brazil

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ABSTRACT

Given a bounded domain Ω in $\mathbb{R}^2$, we investigate the scalar Chern–Simons equation

$$-\Delta u + e^u(e^u - 1) = \mu \quad \text{in } \Omega,$$

in cases where there is no solution for a given nonnegative finite measure μ. Approximating μ by a sequence $(\mu_n)_{n\in\mathbb{N}}$ of nonnegative L^1 functions or finite measures for which this equation has a solution, we show that the sequence of solutions $(u_n)_{n\in\mathbb{N}}$ of the Dirichlet problem converges to the solution with largest possible datum $\mu^\# \leq \mu$ and we derive an explicit formula of $\mu^\#$ in terms of μ. The counterpart for the Chern-Simons system with datum (μ, ν) behaves differently and the conclusion depends on how much the measures μ and ν charge singletons.

1. Introduction and main results

In this paper we investigate a question concerning convergence and stability of solutions of the scalar Chern–Simons problem

$$\begin{cases} -\Delta u + e^u(e^u - 1) = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^2$ is a smooth bounded domain and μ is a finite Borel measure – equivalently a Radon measure – in Ω. By a solution of (1.1), we mean a function $u \in W_0^{1,1}(\Omega)$ such that $e^u(e^u - 1) \in L^1(\Omega)$ and satisfying the equation in the sense of distributions.

Using for instance a minimization argument in $W_0^{1,2}(\Omega)$, one shows that the scalar Chern–Simons equation always has a solution with datum $\mu \in L^p(\Omega)$ for any $1 < p \leq \infty$ [1, Chapter 2]. Existence in the case of datum $\mu \in L^1(\Omega)$ can be obtained by approximation using L^∞ data [2, Corollary 12]; [1, Chapter 3].

The case of nonlinear Dirichlet problems with measure data is more subtle. This issue has been discovered by Bénilan and Brezis [3–5] in a pioneering work concerning polynomial nonlinearities in dimension greater than 2.

The case of exponential nonlinearities in dimension 2 has been investigated by Vázquez [6]. For instance, if $\mu = \alpha\delta_a$ for some $a \in \Omega$, then for every $\alpha > 2\pi$ the Dirichlet problem (1.1) has no solution with datum μ. The counterexample above

* Corresponding author.
E-mail address: presoto@dm.ufscar.br (A.E. Presoto).

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gives the only possible obstruction in the case of exponential nonlinearities: μ is a good measure – that is the Dirichlet problem (1.1) has a solution – if and only if for every $x \in \Omega$,

$$\mu(\{x\}) \leq 2\pi.$$

We want to understand what happens when one forces the Dirichlet problem to have a solution when no solution is available. For instance, if μ is a measure for which (1.1) has no solution, then one could approximate μ by a sequence $(\rho_n * \mu)_{n\in\mathbb{N}}$ of convolutions of μ – for which we know the Dirichlet problem has a solution – and then investigate the limit of the sequence of solutions $(u_n)_{n\in\mathbb{N}}$.

This program has been proposed and implemented by Brezis, Marcus and Ponce [7] in the case where μ is approximated via convolution. They have proved that for any sequence of nonnegative mollifiers $(\rho_n)_{n\in\mathbb{N}}$, if u_n satisfies

$$\begin{cases} -\Delta u_n + e^{u_n}(e^{u_n} - 1) = \rho_n * \mu & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases}$$

then the sequence $(u_n)_{n\in\mathbb{N}}$ converges in $L^1(\Omega)$ to the largest subsolution u^* of the scalar Chern–Simons problem with datum μ [7, Theorem 4.11].

The result in [7] concerns more general convex nonlinearities and holds in any dimension, but strongly relies on the fact that the approximating sequence $(\rho_n * \mu)_{n\in\mathbb{N}}$ is constructed via convolution of μ [7, Example 4.1].

Our first result shows that for the Chern–Simons equation the conclusion of Brezis, Marcus and Ponce is always true regardless of the sequences of functions – or even measures – $(\mu_n)_{n\in\mathbb{N}}$ converging to μ.

Theorem 1.1. *Let $(\mu_n)_{n\in\mathbb{N}}$ be a nonnegative sequence of measures in Ω such that for every $n \in \mathbb{N}$ and for every $x \in \Omega$,*

$$\mu_n(\{x\}) \leq 2\pi$$

and let u_n satisfy the scalar Chern–Simons problem

$$\begin{cases} -\Delta u_n + e^{u_n}(e^{u_n} - 1) = \mu_n & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases}$$

If the sequence $(\mu_n)_{n\in\mathbb{N}}$ converges to a measure μ in the sense of measures in Ω, then the sequence $(u_n)_{n\in\mathbb{N}}$ converges in $L^1(\Omega)$ to the solution of the scalar Chern–Simons problem with datum $\mu^\#$, where $\mu^\#$ is the largest measure less than or equal to μ such that for every $x \in \Omega$,

$$\mu^\#(\{x\}) \leq 2\pi.$$

A sequence $(\mu_n)_{n\in\mathbb{N}}$ converges weakly to μ in the sense of measures in Ω, if for every continuous function $\zeta : \overline{\Omega} \to \mathbb{R}$ such that $\zeta = 0$ on $\partial\Omega$,

$$\lim_{n\to\infty} \int_\Omega \zeta \, d\mu_n = \int_\Omega \zeta \, d\mu.$$

We denote this convergence by $\mu_n \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}(\Omega)$, where $\mathcal{M}(\Omega)$ is the vector space of (finite) measures in Ω equipped with the norm

$$\|\mu\|_{\mathcal{M}(\Omega)} = |\mu|(\Omega) = \int_\Omega d|\mu|.$$

Applying Theorem 1.1 we deduce an explicit formula of $\mu^\#$ in terms of μ. Indeed, if we write μ as a sum of nonatomic part $\overline{\mu}$ and an atomic part

$$\mu = \overline{\mu} + \sum_{i=0}^{\infty} \alpha_i \delta_{a_i},$$

where $\alpha_i \geq 0$ and the points a_i are distinct, then

$$\mu^\# = \overline{\mu} + \sum_{i=0}^{\infty} \min\{\alpha_i, 2\pi\} \delta_{a_i}.$$

Since μ is a finite measure, there can only be finitely many indices i such that $\alpha_i > 2\pi$. In particular, the measure $\mu - \mu^\#$ is supported in a finite set and for every $a \in \Omega$,

$$\mu^\#(\{a\}) = \min\{\mu(\{a\}), 2\pi\}.$$

We may recover the result of Brezis, Marcus and Ponce using their notion of reduced measure μ^*. By definition, the reduced measure is the unique locally finite measure in Ω such that

$$\mu^* = -\Delta u^* + e^{u^*}(e^{u^*} - 1)$$