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A parallel two-level finite element variational multiscale method for the Navier–Stokes equations☆

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ABSTRACT

A combination method of two-grid discretization approach with a recent finite element variational multiscale algorithm for simulation of the incompressible Navier–Stokes equations is proposed and analyzed. The method consists of a global small-scale nonlinear Navier–Stokes problem on a coarse grid and local linearized residual problems in overlapped fine grid subdomains, where the numerical form of the Navier–Stokes equations on the coarse grid is stabilized by a stabilization term based on two local Gauss integrations at element level and defined by the difference between a consistent and an under-integrated matrix involving the gradient of velocity. By the technical tool of local a priori estimate for the finite element solution, error bounds of the discrete solution are estimated. Algorithmic parameter scalings are derived. Numerical tests are also given to verify the theoretical predictions and demonstrate the effectiveness of the method.

1. Introduction

Large-scale computational fluid problems need large computing resources that may only be provided by high performance parallel computers or a cluster of workstations. Consequently, with the development of technology for parallel computing, parallel computation becomes a main tool for simulation of large-scale fluid flows. In such parallel computing, parallel numerical algorithms play a key role both in ensuring the accuracy of the computed approximate solutions and in exploiting the full potential of computational power of the parallel computer in use. In general, a computational fluid dynamics model is based on the solution of the Navier–Stokes equations and its discretization scheme such as finite element methods, finite volume methods and finite difference methods. Therefore, the development of efficient parallel algorithms for the Navier–Stokes equations attracts much attention in computational fluid dynamics community (see, e.g., [1–6]).

Recently, based on the philosophy that phenomena of different scales should be treated by different tools and following the idea of local and parallel finite element computations of Xu and Zhou [7,8], some local and parallel algorithms for simulation of the Navier–Stokes equations were proposed and analyzed in [9–15]. Motivated by the fact that for a finite element solution to the Navier–Stokes equations, its global behavior is mostly dominated by the low frequency components and, on the contrary, the local behavior is basically affected by high frequency components, these algorithms first solve the fully nonlinear Navier–Stokes equations on a coarse grid by the standard finite element method to approximate the low frequencies, and then solve the resultant linearized residual equations (which mostly contains the high frequencies) in overlapped fine grid subdomains to correct the coarse grid solution locally or in parallel.

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However, due to the fact that a fully nonlinear Navier–Stokes problem needs to be solved by the standard finite element method on a coarse grid, it is challenging for these local and parallel algorithms to simulate high Reynolds number flows. It is well known that high Reynolds number flows have a wide spectrum of scales which may span several orders of magnitude. If the mesh of the underlying discretization of the Navier–Stokes equations is not fine enough to represent all the scales of the flow, spurious oscillations may occur in numerical simulations leading to an unstable or inaccurate numerical solution. What is worse, the iterative methods used to solve the nonlinear system may fail to converge and, consequently, cannot yield a numerical solution at all. This was well documented in literature. For example, for the well-known benchmark 2D lid-driven cavity flow problem, Layton et al. [16] reported that at $\mathrm{Re} = 3200$ and on a 31×31 grid mesh, the standard finite element method combined with a continuation method fails to work, while on a 81×81 uniform grid mesh, Wang [17] was just able to compute a numerical solution at Reynolds numbers up to $\mathrm{Re} = 5000$ by using the standard finite element method. Even for very simple Navier–Stokes problems with known analytical solution, He et al. [18] and Shang [19] showed that to ensure a stable finite element solution at high Reynolds numbers, the underlying mesh size should be small enough. Therefore, to apply these local and parallel algorithms to the simulation of high Reynolds number flows, stabilization techniques are essential.

In this paper, we study a combination method of the above mentioned approach to local and parallel finite element computations with a recent variational multiscale method based on two local Gauss integrations [20]. This combination is particularly efficient and combines the best algorithmic features of each. Specifically, we first solve a nonlinear Navier–Stokes problem with a stabilization term on a coarse grid, and then solve a linearized problem in overlapped fine grid subdomains to correct the solution, where the stabilization term is based on two local Gauss integrations and defined by the difference between a consistent and an under-integrated matrix involving the gradient of velocity. The method can be casted into the framework of variational multiscale method. However, compared to the common variational multiscale methods (cf. [21–24]) where the large scales are defined by projections into appropriate function spaces, this method avoids the construction of any projection operators and does not need extra storage.

It is noted that the parallel approach of the method developed in this paper is similar to that of [9]; however, there is significant novelty for two reasons. First, by adding a variational multiscale stabilization term to the coarse grid problem, the method of this paper aims at the simulation of high Reynolds number flows, while that of [9] is limited to the case of low and moderate Reynolds number flows (due to it uses the standard finite element method to solve the nonlinear Navier–Stokes equations on the coarse grid which fail to work for high Reynolds number flows as mentioned above). Specifically, for laminar flows, on one hand, the stabilization used in our method does not degrade rates of convergence and accuracy of the approximate solution compared to the method of [9] (see Section 5.1); on the other hand, our method is able to simulate high Reynolds number flows for which the method of [9] fails to work (please see Section 5.2). As a result, this paper can be considered as a sequel of the work in [9]. Second, we derive detailed error estimates for the finite element variational multiscale method based on two local Gauss integrations. Although our main motivation is to use them to devise and analyze our new combination method, these error estimates for the finite element variational multiscale method are theoretically interesting in their own right; please see Section 3 for detailed information.

An outline of the paper is as follows. In the next section, the functional setting of the Navier–Stokes equations is given and some mathematical preliminaries for the numerical analysis are provided. In Section 3, the finite element variational multiscale method based on two local Gauss integrations and its error estimates are derived. Local and parallel finite element variational multiscale algorithms together with their priori error estimates for the Navier–Stokes equations are presented in Section 4. Section 5 is devoted to numerical tests which verify the theoretical analyses and demonstrate the effectiveness of the proposed parallel method. Finally, the article is concluded in Section 6.

2. Preliminaries

Let Ω be a bounded domain with Lipschitz-continuous boundary $\partial\Omega$ in $\mathbb{R}^n$ ($n = 2$ or 3). As usual, for a nonnegative integer k, we denote by $H^k(\Omega)$ the Sobolev space of functions with square integrable distribution derivatives up to order k in Ω, equipped with the standard norm $\|\cdot\|_{k,\Omega}$, while denote by $H_0^1(\Omega)$ the closed subspace of $H^1(\Omega)$ consisting of functions with zero trace on $\partial\Omega$; see, e.g., [25,26]. Moreover, for a subdomain $\Omega_0 \subset \Omega$, we view $H_0^1(\Omega_0)$ as a subspace of $H_0^1(\Omega)$ by extending the functions in $H_0^1(\Omega_0)$ to be functions in $H_0^1(\Omega)$ with zero outside of Ω_0. For $D \subset \Omega_0 \subset \Omega$, we use the notation $D \subset\subset \Omega_0$ to mean that $\mathrm{dist}(\partial D \setminus \partial\Omega, \partial\Omega_0 \setminus \partial\Omega) > 0$. Throughout this paper, we shall use the letter c or C (with or without subscripts) to denote a generic positive constant which is independent of the mesh parameter and may take on different values on different occurrences.

2.1. The Navier–Stokes equations

We consider the following incompressible Navier–Stokes equations:

$$-\nu\Delta u + (u\cdot\nabla)u + \nabla p = f, \quad \text{in } \Omega, \tag{2.1}$$

$$\mathrm{div}\, u = 0, \quad \text{in } \Omega, \tag{2.2}$$

$$u = 0, \quad \text{on } \partial\Omega. \tag{2.3}$$