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Existence results for non-local operators of elliptic type

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ABSTRACT

In this paper, we investigate the existence of solutions for equations driven by a non-local integrodifferential operator with homogeneous Dirichlet boundary conditions. We make use of homological linking and Morse theory.

1. Introduction

This paper is concerned with the following problem

$$\begin{cases} \mathcal{L}_K u + \lambda u + f(x,u) = 0 & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ is a bounded domain with smooth boundary $\partial\Omega$, $\mathcal{L}_K$ is the non-local operator defined by:

$$\mathcal{L}_K u(x) = \int_{\mathbb{R}^n} (u(x+y) + u(x-y) - 2u(x))K(y)dy, \quad x \in \mathbb{R}^n.$$

Here $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ is a function such that

$$mK \in L^1(\mathbb{R}^n), \quad \text{where } m(x) = \min\{|x|^2, 1\}; \tag{1.2}$$

there exist $\theta > 0$ and $s \in (0, 1)$ such that

$$K(x) \geq \theta |x|^{-(n+2s)} \quad \text{for any } x \in \mathbb{R}^n \setminus \{0\}; \tag{1.3}$$

$$K(x) = K(-x) \quad \forall x \in \mathbb{R}^n \setminus \{0\}. \tag{1.4}$$

A typical example for K is given by $K(x) = |x|^{-(n+2s)}$. In this case $\mathcal{L}_K$ is the fractional Laplace operator $-(-\Delta)^s$ defined as

$$-(-\Delta)^s u(x) = \int_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2s}} dy, \quad x \in \mathbb{R}^n,$$

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where $s \in (0, 1)$ and $n > 2s$. The fractional Laplacian $-(-\Delta)^s$ is a classical linear integrodifferential operator of order $2s$ which gives the standard Laplacian when $s = 1$.

Let X denotes the linear space of Lebesgue measurable functions from $\mathbb{R}^n$ to $\mathbb{R}$ such that the restriction to Ω of any function g in X belongs to $L^2(\Omega)$ and

$$\text{the map } (x, y) \to (g(x) - g(y))\sqrt{K(x-y)} \text{ is in } L^2(\mathbb{R}^{2n} \setminus (\mathcal{C}\Omega \times \mathcal{C}\Omega), dxdy),$$

where $\mathcal{C}\Omega := \mathbb{R}^n \setminus \Omega$. Moreover,

$$X_0 = \{g \in X : g = 0 \text{ a. e. in } \mathbb{R}^n \setminus \Omega\}.$$

We say that $u \in X_0$ is a weak solution of problem (1.1), if u satisfies

$$\begin{cases} \int_{\mathbb{R}^{2n}} (u(x) - u(y))(\varphi(x) - \varphi(y))K(x-y)dxdy - \lambda \int_{\Omega} u(x)\varphi(x)dx \\ \quad = \int_{\Omega} f(x, u(x))\varphi(x)dx \quad \forall \varphi \in X_0 \\ u \in X_0. \end{cases} \tag{1.5}$$

Recently, Servadei and Valdinoci [1] have investigated the existence of nontrivial weak solutions for non-local problem (1.1) by using functional analytical setting. The fractional Laplacian and non-local operators of elliptic type arises in both pure mathematical research and concrete applications, since these operators occur in a quite natural way in many different contexts and instances such as the minimal surfaces [2], the thin obstacle problem [3,4], Markov processes [5], phase transitions [6] and fractional quantum mechanics [7]. For an elementary introduction to this topic, see [8] and the references therein. Lately, some elliptic boundary problems driven by the non-local integrodifferential operator $\mathcal{L}_K$ are studied in [9–11].

Denote by $0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots$ the eigenvalues of the following non-local eigenvalue problem

$$\begin{cases} -\mathcal{L}_K u = \lambda u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{1.6}$$

In [9], by virtue of the mountain-pass theorem, Servadei and Valdinoci presented one non-zero weak solution of (1.1) for $\lambda < \lambda_1$, and one non-zero weak solution of (1.1) for $\lambda \geq \lambda_1$ was given by using the linking theorem.

Motivated by the papers [12,9], we obtain some multiplicity results on the existence of nontrivial weak solutions for problem (1.1) due to homological linking and Morse theory. It is worth mentioning that in [9] Servadei and Valdinoci have proved the existence of weak solutions for any value of the parameter λ, while in this paper we study the existence and multiplicity results for problem (1.1) for parameter λ which is sufficiently close to the eigenvalues λ_k. On the other hand, our approach here is different from that of [9].

The paper is organized as follows. In Section 2, some notions and lemmas are given. In Section 3, some existence results of one weak solution and two weak solutions for problem (1.1) are obtained by using homological linking and Morse theory.

2. Preliminaries

For $c \in \mathbb{R}$, set

$$I^c = \{u \in E | I(u) \leq c\}, \qquad \mathcal{K}_c = \{u \in E | I'(u) = 0, I(u) = c\}.$$

We say a functional $I \in C^1(E, \mathbb{R})$ has a local linking structure at 0 with respect to a direct sum decomposition $E = Y \bigoplus Z$ if there is an $r > 0$ such that

$$I(u) \leq 0 \quad \text{for } u \in Y \text{ with } \|u\| \leq r, \qquad I(u) > 0 \quad \text{for } u \in Z \text{ with } 0 < \|u\| \leq r.$$

Recall that the q-th critical group of I at its isolated critical point u is defined as

$$C_q(I, u) := H_q(I^c \cap U, I^c \cap U \setminus \{u\}).$$

Here $c = I(u)$ and $H_q(A, B)$ is the q-th relative singular homology group of the topological pair (A, B) with coefficients in a field $\mathbb{F}$. The critical groups at infinity were introduced by Bartsch and Li [13] as

$$C_q(I, \infty) := H_q(E, I^\alpha). \tag{2.1}$$

Note that by the deformation lemma, the right-hand side of (2.1) does not depend on the choice of α. The reader is referred to [14] for more details on Morse theory.

In the sequel we set $Q = \mathbb{R}^{2n} \setminus \mathcal{O}$, where

$$\mathcal{O} = (\mathcal{C}\Omega \times \mathcal{C}\Omega) \subset \mathbb{R}^{2n},$$

and $\mathcal{C}\Omega = \mathbb{R}^n \setminus \Omega$.