Nonlinear Analysis 78 (2013) 47–61

Contents lists available at SciVerse ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Global existence of solutions to the initial-boundary value problem of conservation law with degenerate diffusion term

Jiao Chen, Yachun Li *, Weike Wang

Department of Mathematics & Key Lab of Scientific and Engineering Computing (MOE), Shanghai Jiao Tong University, Shanghai 200240, China

ARTICLE INFO

Article history:
Received 12 November 2011
Accepted 21 September 2012
Communicated by Enzo Mitidieri

MSC:
35K65

Keywords:
Classical solutions
Global existence
Conservation law
Degenerate diffusion
Frequency decomposition method
Energy method
Green's function method

ABSTRACT

In this paper we explore the classical solutions to the conservation law with degenerate diffusion term ($u_t - \Delta_{x'} u = \operatorname{div} f(u)$, $x \in \Omega \subset \mathbb{R}^n$, $t > 0$, with $x = (x_1, x')$). We establish the global existence and exponential decay estimates to the solutions of the initial boundary value problem in domain $\Omega = \mathbb{R} \times \prod_{i=2}^{n}(0, L_i)$. Meanwhile, to clarify the viscous effect of the degenerate diffusion term, we also investigate the classical solutions to the Cauchy problem of the modified equation $u_t - \Delta_{x'} u = (1 - \chi(D))\operatorname{div} f(u)$, $x \in \mathbb{R}^n$, $t > 0$, with $\chi(D)$ a Fourier multiplier operator, we use the frequency decomposition method to establish the global existence and the polynomial decay estimates.

1. Introduction

In this paper, we study the global existence and decay estimates for solutions of the following conservation law with degenerate diffusion:

$$u_t - \Delta_{x'} u = \operatorname{div} f(u), \quad x \in \Omega,\ t > 0, \tag{1.1}$$

where $f = f(u)$ is a given vector function of $u \in \mathbb{R}$ in the form of $f = (f_1, \dots, f_n)$. Here each $f_i(u) = O(|u|^{1+\theta})$ is sufficiently smooth and $\theta \geq 1$ is an integer. Hereafter, we also denote $(f_2(u), \dots, f_n(u))$ by $g(u)$ for convenience, then $f(u) = (f_1(u), g(u))$.

Eq. (1.1) is a special case of the general degenerate parabolic–hyperbolic equation

$$u_t + \operatorname{div} f(u) - \nabla \cdot (A(u)\nabla u) = 0, \qquad (t, x) \in [0, T) \times \mathbb{R}^n, \tag{1.2}$$

where $f = (f_1, \dots, f_n)$ is a given vector-valued flux function, and $A = (a_{ij})$ is a given symmetric matrix-valued diffusion function of the form

$$\begin{cases} A(u) = \sigma(u)\sigma(u)^{\top} \geq 0, \\ \sigma(u) \in \mathbb{R}^{n \times K}, \quad 1 \leq K \leq n. \end{cases} \tag{1.3}$$

* Corresponding author.
E-mail addresses: jiaojiaochen@sjtu.edu.cn (J. Chen), ycli@sjtu.edu.cn (Y. Li), wkwang@sjtu.edu.cn (W. Wang).

0362-546X/$ – see front matter © 2012 Elsevier Ltd. All rights reserved.
doi:10.1016/j.na.2012.09.016

The nonnegativity requirement in (1.3) means that for all $u \in \mathbb{R}$

$$\sum_{i,j=1}^{n} a_{ij}(u)\lambda_i\lambda_j \geq 0, \quad \forall\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n.$$

When σ takes the form

$$\sigma(u) = \nu(u)I$$

for some scalar function $\nu(u)$, (1.3) is called isotropic, otherwise it is called anisotropic. A special case of the anisotropic is quasi-isotropic with a diagonal diffusion matrix: $a_{ij} = 0$ for $i \neq j$.

Eq. (1.2) models many physical phenomena, such as two phase flows in a porous medium (cf. [1] and the references cited therein), and the sedimentation-consolidation process (cf. [2] and reference cited therein). There are two extremal cases of (1.2), one is the hyperbolic scalar conservation law

$$u_t = \mathrm{div} f(u) \tag{1.4}$$

when the diffusion is fully degenerate. It is well known that the solution will blow up in finite time [3]. The other one is the following scalar conservation law with non-degenerate viscosity

$$u_t - \Delta u = \mathrm{div} f(u), \tag{1.5}$$

about which there are many results on the existence and stability. For the case of small perturbation, Kawashima and Matsumura [4] obtained that the solution of (1.5) approaches the traveling wave solution at the rate $t^{-\gamma}$ (for some $\gamma > 0$) as $t \to \infty$ in one-dimensional space. Goodman [5] proved nonlinear stability in L^2 of planar shock front solutions for Eq. (1.5) in two or more space dimensions. Goodman and Miller [6] investigated nonlinear stability in L^1 of planar shock front solutions to a viscous conservation law in two spatial dimensions and obtained an expression for the asymptotic form of small perturbations. For the case of large perturbation, Kotlow [7] studied the existence of the solution for Eq. (1.5) in 1971. Deng and Wang [8] obtained the pointwise estimate of large perturbation around viscous shock for (1.5) in one dimension by full use of decaying properties of solutions and the weighted energy method.

As for Eq. (1.2), because of the degeneracy of the diffusion, people are focused on the study of discontinuous solutions. There are many different kinds of solution definitions and corresponding results in these solution classes such as weak solutions, entropy solutions, renormalized solutions, dissipative solution, and kinetic solutions (see [9–20], and the references cited therein). However no reference on the well-posedness of smooth solutions can be found so far.

Eq. (1.1) is actually quasi-isotropic with the diffusion matrix $A(u) = \mathrm{diag}\{0, 1, \ldots, 1\}$ which is degenerate in the x_1 direction. The main difficulty of this equation is that there are no viscous effect in the x_1 direction, so the commonly used methods in the parabolic equation or even in the hyperbolic–parabolic equation cannot be used here. In our opinion, solutions of this equation will generally blow up in finite time, the existence of the global solution to the Cauchy problem in $\mathbb{R}^n$ with even small initial data is still an open problem, we cannot find a good approach to deal with it. However, for an initial boundary value problem in $\Omega = \mathbb{R} \times U$ where $U = \prod_{i=2}^{n}(0, L_i)$ with the following data

$$u(x, 0) = u_0(x), \tag{1.6}$$

$$u(x, t)|_{\partial U} = 0, \tag{1.7}$$

we can get the global classical solution when initial perturbation is small. This is because we have a Poincaré-like inequality in this case due to the boundedness of the domain along the x' direction and the homogeneous boundary conditions. Thanks to this inequality, the diffusion term $\Delta_{x'} u$ in the x' direction can work as a damping term in the x_1 direction to a certain extent.

On the other hand, we still have a Poincaré-like inequality for high frequency case and thus the diffusion $\Delta_{x'} u$ can still work as a damping term in the x_1 direction no matter whether the domain is bounded or unbounded. That is to say, for the Cauchy problem, if the low frequency part does not appear in the nonlinear term, we can still have a damping effect by the Poincaré-like inequality and then obtain the global classical solution with small perturbation. To make this point clearer, we consider the Cauchy problem of the following artificially modified equation

$$u_t - \Delta_{x'} u = (1 - \chi(D))\mathrm{div} f(u), \quad x \in \mathbb{R}^n, \ t > 0, \tag{1.8}$$

where $f(\cdot) = (f_1(\cdot), g(\cdot))$ is the same as that in (1.1). Here the nonlinear term is affected by a nonlocal operator $1 - \chi(D)$. $\chi(D)$ is a Fourier multiplier operator with the symbol

$$\chi(\xi) = \begin{cases} 1, & |\xi'| \leq \eta, \\ 0, & |\xi'| > \eta, \end{cases} \tag{1.9}$$

which is a cut-off function for some fixed constant $\eta > 0$. From the point of view of frequency space, the effect of the nonlocal operator is to cut off the low frequency part of $\mathrm{div} f(u)$, namely, the low frequency wave has no nonlinear effect, which shares the same mechanism as the fact that operator $\Delta_{x'}$ has the smallest positive eigenvalue.