Nonlinear Analysis 78 (2013) 79–85

Contents lists available at SciVerse ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Local Hölder regularity of the gradients for the elliptic $p(x)$-Laplacian equation

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A R T I C L E I N F O

Article history:
Received 17 September 2012
Accepted 24 September 2012
Communicated by Enzo Mitidieri

MSC:
35J60
35J70

Keywords:
$C^{1,\alpha}$
Hölder
Regularity
Gradient
Divergence
Elliptic
$p(x)$-Laplacian

A B S T R A C T

In this paper we obtain the interior Hölder regularity of the gradients of weak solutions for the elliptic $p(x)$-Laplacian equation

$$\operatorname{div}\left((A\nabla u\cdot\nabla u)^{\frac{p(x)-2}{2}}A\nabla u\right)=\operatorname{div}\left(|\mathbf{f}|^{p(x)-2}\mathbf{f}\right),$$

under some proper assumptions on the Hölder continuous functions p, $\mathbf{f}$ and A.

1. Introduction

In this paper we mainly study the interior Hölder regularity of the gradients of weak solutions for the following elliptic $p(x)$-Laplacian equation

$$\operatorname{div}\left((A\nabla u\cdot\nabla u)^{\frac{p(x)-2}{2}}A\nabla u\right)=\operatorname{div}\left(|\mathbf{f}|^{p(x)-2}\mathbf{f}\right)\quad\text{in }\Omega, \tag{1.1}$$

where Ω is an open bounded domain in $\mathbb{R}^n$, $\mathbf{f}=(f^1,\dots,f^n)$ is a given vector field and $A=\{a_{ij}\}$ is a symmetric matrix with measurable coefficients satisfying

$$1<\gamma_1=\inf_{\Omega}p(x)\le\sup_{\Omega}p(x)=\gamma_2<\infty, \tag{1.2}$$

$$\Lambda^{-1}|\xi|^2\le A(x)\xi\cdot\xi\le\Lambda|\xi|^2 \tag{1.3}$$

and

$$p(x)\in C^{0,\alpha_1}_{loc}(\Omega),\qquad f^i(x)\in C^{0,\alpha_2}_{loc}(\Omega)\quad\text{and}\quad a_{ij}(x)\in C^{0,\alpha_3}_{loc}(\Omega) \tag{1.4}$$

for any $\xi, x\in\mathbb{R}^n$ and $1\le i,j\le n$, where $\alpha_1,\alpha_2,\alpha_3,\Lambda>0$ are positive constants.

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0362-546X/$ – see front matter
doi:10.1016/j.na.2012.09.017

When $p(x)$ is a constant, many authors [1–7] studied the regularity for weak solutions of quasilinear elliptic equations of p-Laplacian type. When $p(x)$ is not a constant, such elliptic problems (1.1) appear in mathematical models of various physical phenomena, such as the electro-rheological fluids (see, e.g., [8–10]). Especially when $A = I$ and $\mathbf{f} = 0$, (1.1) is reduced to the $p(x)$-Laplacian elliptic equation

$$\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) = 0 \quad \text{in } \Omega, \tag{1.5}$$

which can be derived from the variational problem

$$\Phi(u) = \min_{v|_{\partial\Omega}=\varphi} \Phi(v) =: \min_{v|_{\partial\Omega}=\varphi} \int_\Omega \frac{1}{p(x)}|\nabla v|^{p(x)} dx.$$

There have been many investigations [11–13] on Hölder estimates for the $p(x)$-Laplacian elliptic equation (1.5). Recently, Challal and Lyaghfouri [14] obtained the local L^∞ estimates of $|\nabla u|^{p(x)}$ for the weak solutions of (1.5). Moreover, Acerbi and Mingione [15] have proved that

$$|\mathbf{f}|^{p(x)} \in L^q_{loc}(\Omega) \Longrightarrow |\nabla u|^{p(x)} \in L^q_{loc}(\Omega) \quad \text{for any } q > 1$$

for weak solutions of (1.1) under some assumptions on $p(x)$.

We denote by $L^{p(x)}(\Omega)$ the variable exponent Lebesgue–Sobolev spaces

$$L^{p(x)}(\Omega) = \left\{ f : \Omega \to \mathbb{R} \mid f \text{ is measurable and } \int_\Omega |f|^{p(x)}\, dx < \infty \right\} \tag{1.6}$$

with the Luxemburg type norm

$$\|f\|_{L^{p(x)}(\Omega)} = \inf \left\{ \lambda > 0 : \int_\Omega \left|\frac{f}{\lambda}\right|^{p(x)} dx \le 1 \right\}. \tag{1.7}$$

Furthermore, we define

$$W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega) \right\} \tag{1.8}$$

with the norm

$$\|u\|_{W^{1,p(x)}(\Omega)} = \|u\|_{L^{p(x)}(\Omega)} + \|\nabla u\|_{L^{p(x)}(\Omega)}. \tag{1.9}$$

By $W_0^{1,p(x)}(\Omega)$ we denote the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. Actually, the $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ spaces are Banach spaces. There have been many investigations (see for example [16–21]) on properties of such variable exponent Sobolev spaces.

As usual, the solutions of (1.1) are taken in a weak sense. We now state the definition of weak solutions.

Definition 1.1. Assume that $\mathbf{f} \in L^{p(x)}_{loc}(\Omega)$. A function $u \in W^{1,p(x)}_{loc}(\Omega)$ is a local weak solution of (1.1) in Ω if for any $\varphi \in W_0^{1,p(x)}(\Omega)$, we have

$$\int_\Omega (A\nabla u \cdot \nabla u)^{\frac{p(x)-2}{2}} A\nabla u \cdot \nabla\varphi\, dx = \int_\Omega |\mathbf{f}|^{p(x)-2}\mathbf{f} \cdot \nabla\varphi\, dx.$$

Now let us state the main result of this work.

Theorem 1.2. *If u is a local weak solution of problem (1.1) under the assumptions (1.2)–(1.4), then ∇u is locally Hölder continuous.*

2. Proof of the main result

In this section we shall finish the proof of Theorem 1.2. We first recall the following reverse Hölder inequality.

Lemma 2.1 (*See [15, Lemma 5]*). *If u is a local weak solution of problem (1.1) under the assumptions (1.2)–(1.4), then there exist positive constants σ_0, $R_0 < 1$, C, depending on n, γ_1, γ_2, Λ, such that*

$$\fint_{B_{R/2}} |\nabla u|^{p(x)(1+\sigma)}\, dx \le C\left(\fint_{B_R} |\nabla u|^{p(x)}\, dx\right)^{1+\sigma} + C\left(\fint_{B_R} 1 + |\mathbf{f}|^{p(x)(1+\sigma)}\, dx\right)$$

holds for any $R \le R_0$ and $\sigma \le \sigma_0$.