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Existence, uniqueness and stability for a class of third-order dissipative problems depending on time

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ABSTRACT

We prove new results regarding the existence, uniqueness, (eventual) boundedness, (total) stability and attractivity of the solutions of a class of initial–boundary-value problems characterized by a quasi-linear third-order equation which may contain time-dependent coefficients. The class includes equations arising in superconductor theory and in the theory of viscoelastic materials. In the proof we use a Liapunov functional V depending on two parameters, which we adapt to the characteristics of the problem.

1. Introduction

As is known, dealing with (in)stability in non-autonomous problems in general requires careful generalizations of criteria and methods valid for autonomous problems, even in linear, finite-dimensional systems (see e.g. [1–5]). The Liapunov direct method in its general formulation applies to non-autonomous (as well as to autonomous) systems, but the construction of Liapunov functions is more complicated.

In this paper we consider a class of non-autonomous initial–boundary-value problems having a number of different physical applications and prove new results regarding the existence, uniqueness, boundedness, stability and attractivity of their solutions; the problems have the form

$$\begin{cases} L\varphi = h(x,t,\Phi), L(t) := \partial_t^2 + a\partial_t - C(t)\partial_x^2 - \varepsilon(t)\partial_x^2\partial_t \quad x \in]0,\pi[,\ t > t_0, \\ \varphi(0,t) = \phi_0(t), \qquad \varphi(\pi,t) = \phi_\pi(t), \end{cases} \tag{1.1}$$

$$\varphi(x,t_0) = \varphi_0(x), \qquad \varphi_t(x,t_0) = \varphi_1(x). \tag{1.2}$$

Here $\Phi := (\varphi, \varphi_x, \varphi_t)$, $t_0 \geq 0$, $\varepsilon \in C^2(I,I)$, $C \in C^1(I,\mathbb{R}^+)$ (with $I := [0,\infty[$) are functions of t, with $C(t) \geq \overline{C} = \text{const} > 0$; $a = \text{const}$, $\varepsilon(t) \geq 0$, $h \in C([0,\pi] \times I \times \mathbb{R}^3)$; $\phi_0, \phi_\pi \in C^2(I)$, $u_0, u_1 \in C^2([0,\pi])$ are assigned and fulfill the consistency conditions

$$\phi_0(t_0) = \varphi_0(0), \qquad \dot{\phi}_0(t_0) = \varphi_1(0), \qquad \phi_\pi(t_0) = \varphi_0(\pi), \qquad \dot{\phi}_\pi(t_0) = \varphi_1(\pi). \tag{1.3}$$

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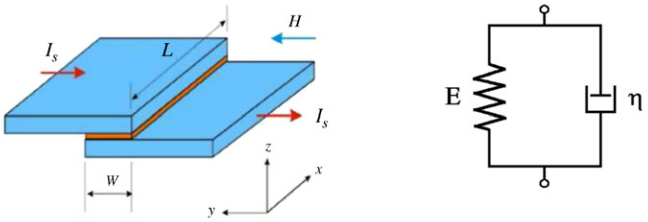

Fig. 1. Josephson junction (left) and schematic representation of a Voigt material (right). W, L are the width and length of the JJ; I_s, H are the total superconducting current and the external magnetic field.

We wish to compare problem (1.1) + (1.2) to the perturbed one

$$\begin{cases} Lw = h(x,t,W) + k(x,t), & x \in]0,\pi[, \; t > t_0, \\ w(0,t) = \phi_0(t) + w_0(t), & w(\pi,t) = \phi_\pi(t) + w_\pi(t), \end{cases} \tag{1.4}$$

$$w(x,t_0) = \varphi_0(x) + w_0(x), \qquad w_t(x,t_0) = \varphi_1(x) + w_1(x) \tag{1.5}$$

where $W := (w, w_x, w_t)$, $k \in C([0,\pi] \times I)$, $w_0, w_\pi \in C^2(I)$, $w_0, w_1 \in C^2([0,\pi])$ are assigned and fulfill the consistency conditions

$$w_0(t_0) = w_0(0), \qquad \dot{w}_0(t_0) = w_1(0), \qquad w_\pi(t_0) = w_0(\pi), \qquad \dot{w}_\pi(t_0) = w_1(\pi). \tag{1.6}$$

Defining

$$\begin{aligned} &p(x,t) := \frac{x}{\pi} w_\pi(t) + \left(1 - \frac{x}{\pi}\right) w_0(t), \qquad u := w - \varphi - p, \qquad u_0(x) := w_0(x) - p(x,t_0), \\ &u_1(x) := w_1(x) - (\partial_t p)(x,t_0) \qquad f(x,t,U) := h(x,t,U + \Phi + P) - h(x,t,\Phi) - (Lp)(x,t) + k(x,t), \end{aligned} \tag{1.7}$$

where $U := (u, u_x, u_t)$, $P := (p, p_x, p_t)$, we find that u fulfills the initial–boundary-value problem

$$\begin{cases} Lu = f(x,t,U), & x \in]0,\pi[, \; t > t_0, \\ u(0,t) \equiv 0, & u(\pi,t) \equiv 0, \end{cases} \tag{1.8}$$

$$u(x,t_0) = u_0(x), \qquad u_t(x,t_0) = u_1(x). \tag{1.9}$$

u_0, u_1 automatically fulfill the consistency condition $u_0(0) = u_1(0) = u_0(\pi) = u_1(\pi) = 0$. This shows that we can reduce the questions of stability, the attractivity of some φ and the boundedness of $w - \varphi$ to those of the corresponding u around the origin $u \equiv 0$. Note that if $w_0 \equiv w_\pi \equiv 0$, then $p \equiv 0$, $P \equiv 0$, $k \equiv 0$, $f(x,t,0) = 0$, and problem (1.8) admits the null solution, $u(x,t) \equiv 0$. In (1.1), (1.8) the ε-term is dissipative at t if $\varepsilon(t) > 0$, and the a-term is too if $a > 0$.

Physically remarkable examples of problems (1.1) + (1.2) include:

- If $h = b \sin\varphi - \gamma$, with $b, \gamma = \text{const}$, a modified sine–Gordon equation describing the *Josephson effect* [6] in the theory of superconductors, which lies at the base (see e.g. [7]) of a large number of advanced developments both in fundamental research (e.g. macroscopic effects of quantum physics, quantum computation) and in applications to electronic devices (see e.g. Chapters 3–6 in [8]): $\varphi(x,t)$ is the phase difference of the macroscopic wavefunctions of the Bose–Einstein condensate of Cooper pairs in two superconductors separated by a *Josephson junction* (JJ), i.e. a very thin and narrow dielectric strip of finite length (Fig. 1-left), the γ-term is the (external) "bias current" providing energy to the system, the term $a\varphi_t$ is due to the Joule effect of the residual current of single electrons across the JJ, and the term $\varepsilon\varphi_{xxt}$ is due to the surface impedance of the JJ. In the simplest model adopted for describing the JJ, the parameters ε, C are constant (ε is rather small), and $a = 0$; more accurately, a is positive but very small; even more accurately, $h = b\sin\varphi - \gamma - \beta\varphi_t \cos\varphi$ and ε, C, β are positive (ε, β are very small), and depend on the temperature and on the voltage applied to the JJ (see e.g. [9]), which can be controlled and varied with t. Also γ can be varied with t. Finally, if γ, or the temperature [10], or the width of the junction [11,12] is spatially dependent, then new terms linear in φ_x may appear in the equation; in particular if the width is exponentially shaped, the system may be modeled using the choice $h = b\sin\varphi - \gamma - \beta\varphi_t\cos\varphi - \lambda\varphi_x$ ($\lambda = \text{const}$).
- If $a = 0$, $h = h(x,t)$, an equation (see e.g. [13,14]) for the displacement $\varphi(x,t)$ of the section of a rod from its rest position x in a Voigt material: h is the applied density force, $C \equiv c^2 = E/\rho$, $\varepsilon = 1/\rho\eta$, where ρ is the linear density of the rod at rest, E, η are the elastic and viscous constants of the rod, which enter the stress–strain relation $\sigma = E\nu + \partial_t\nu/\eta$, where σ is the stress, and ν is the strain (as is known, a discretized model of the rod is a series of elements consisting of a viscous damper and an elastic spring connected in parallel as shown in Fig. 1-right). Again, E, η may depend on the temperature of the rod, which can be controlled and varied with t.
- Equations used to describe: heat conduction at low temperature φ [15–17], if $\varepsilon = c^2$, $h = 0$; sound propagation in viscous gases [18]; propagation of plane waves in perfect incompressible and electrically conducting fluids [19].