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Potential operators in generalized Hölder spaces on sets in quasi-metric measure spaces without the cancellation property

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ABSTRACT

We consider potential operators of order α over sets Ω in quasi-metric measure spaces and study their mapping properties from the subspace $H_0^\lambda(\Omega)$ of functions in Hölder space $H^\lambda(\Omega)$ vanishing on the boundary of Ω, into the space $H^{\lambda+\alpha}(\Omega)$, if $\lambda+\alpha < 1$. This is proved in a more general setting of generalized Hölder spaces $H^\omega(\Omega)$ with a given dominant ω of modulus of continuity. Statements of such a kind are known in the Euclidean case or in the case of quasimetric measure spaces with the cancellation property. In the general case, when the cancellation property fails, our proofs are based on a special treatment of the potential of a constant function, which in general has a regularity near the boundary $\partial\Omega$ of the type of the α-th power of the distance to $\partial\Omega$. An application to the case of spatial potentials over domains in $\mathbb{R}^n$ and potentials over spherical caps is given.

1. Introduction

Mapping properties of potential operators within the frameworks of Hölder spaces are well studied in the general setting of quasimetric measure spaces (X, ϱ, μ) under the assumption that X satisfies the so called cancellation property; see [1–4]. The well known examples of underlying spaces X with the cancellation property are the whole space $\mathbb{R}^n$ and the sphere $\mathbb{S}^{n-1}$. We also refer to various more precise specifications and/or generalizations of mapping properties of potential operators in these two model cases presented in the papers [5–13].

In cases where the potential of a constant function on X is well defined, the cancellation property means that the potential of a constant is constant. This property was also used in the recent paper [14], where there were admitted potentials of variable order $\alpha(x)$ with possible degeneration: $\alpha(x) = 0$ on a set of measure zero.

The cancellation property is very restrictive in applications: it fails for domains Ω in $\mathbb{R}^n$. In the case of balls in $\mathbb{R}^n$, for instance, the potential of a constant is constant on the boundary, but is not constant in the ball.

In the Euclidean case for instance, statements of the type

$$I_\Omega^\alpha : H^\lambda(\Omega) \to H^{\lambda+\alpha}(\Omega), \quad \Omega \subset \mathbb{R}^n,$$

for the potential operator

$$I_\Omega^\alpha f(x) := \int_\Omega \frac{f(y)\,dy}{|x-y|^{n-\alpha}}$$

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may not be valid for domains, since the potential of a constant has regularity only of order α near the boundary: it behaves in general like $c_1 + c_2[\delta(x)]^\alpha$ near the boundary, where $\delta(x) = \delta(x, \partial\Omega)$ is the distance to the boundary. However, one may expect that there should be a valid statement

$$I^\alpha_\Omega : H^\lambda_0(\Omega) \to H^{\lambda+\alpha}(\Omega) \tag{1.1}$$

for the subspace $H^\lambda_0(\Omega)$ of the Hölder space $H^\lambda(\Omega)$ of functions which vanish at the boundary. Such a mapping is known in the one-dimensional case and goes back to Hardy and Littlewood; see for instance [6, Corollary 1 on p. 56]. A multi-dimensional result of such a kind was recently proved in [15], where in particular the case of uniform domains (Jones domains) was covered. In this paper we develop a similar approach within the framework of general quasimetric measure spaces (X, ϱ, μ) with the growth condition on the measure. We show that a mapping of type (1.1) (and more generally, for spaces of the type $H^\omega(\Omega)$) holds for measurable bounded sets Ω in (X, ϱ, μ) satisfying the so called α-property. Roughly speaking, we can state a result on mapping properties of the potential operator, if we know how the potential operator of the constant, i.e.

$$J_{\Omega,\alpha}(x) = \int_\Omega \frac{d\mu(y)}{\varrho(x, y)^{N-\alpha}}, \quad x \in \Omega, \tag{1.2}$$

where N comes from the growth condition, behaves near the boundary of Ω.

We give the proof of results of such a type in intrinsic terms of the given set $\Omega \subseteq X$. The proof in intrinsic terms allows us to obtain information also about the behaviour of potentials near the boundary $\partial\Omega$ in the cases where $f(x)$ does not vanish at the boundary.

Note that this way was also used in [15] in the case of domains in $\mathbb{R}^n$ and Lebesgue measure, although in this case it is possible to derive just a result of type (1.1) from the estimates of the modulus of continuity of potentials over $\mathbb{R}^n$, obtained in [7], since a function $f \in H^\lambda_0(\Omega)$ may be extended as identical zero outside Ω, which preserves the Hölder behaviour of f. This way was preferred in [15] because it provides information near the boundary, and a derivation of statements even of type (1.1) from [15] is rather artificial: the results in $\mathbb{R}^n$ in [15] were proved in its turn not directly, but by reducing the problem to the case of the unit sphere via the stereographic projection and usage of Fourier–Laplace analysis on the sphere.

The paper is organized as follows. In Section 2 we provide necessary preliminaries related to quasimetric measure spaces (X, ϱ, μ). In Section 3 we study the function $J_{\Omega,\alpha}(x)$, where the main technical statement is Lemma 3.1, and give examples illustrating the behaviour of $J_{\Omega,\alpha}(x)$ near the boundary. In Section 4 we extend the notion of the α-property, introduced in [15] in the Euclidean case, to the general setting. Section 5 contains the main result on the mapping properties. Section 6 contains two applications. The first is related to the case of domains in $\mathbb{R}^n$, where we improve a result from [15] by showing that an arbitrary domain in $\mathbb{R}^n$ satisfies the α-property, introduced in [15]. The second concerns spherical potentials over a spherical cap on the unit sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$, which is inspired by applications studied in [16]. The final Appendix (Appendix) contains some estimates for the case of spherical potentials on a semisphere.

2. Preliminaries on metric measure spaces

Given a set X, a function $\varrho : X \times X \to [0, \infty)$ is called *quasimetric*, if it satisfies the usual metric axioms with the triangle inequality replaced by the *quasi-triangle inequality*

$$\varrho(x, y) \le K[\varrho(x, z) + \varrho(z, y)], \quad K \ge 1 \tag{2.1}$$

where $x, y, z \in X$. We assume that $\varrho(x, y) = \varrho(y, x)$. Let μ be a positive measure on the σ-algebra of subsets of X which contains the d-balls $B(x, r)$. Everywhere in the sequel we suppose that all the balls have finite measure for all $x \in X$ and $r > 0$ and that the space of compactly supported continuous functions is dense in $L^1(X, \mu)$.

We assume that X is closed with respect to the metric ϱ, i.e. every fundamental sequence in X has a limit in X. The boundary $\delta(\Omega)$ of an open set Ω in X is interpreted in the usual sense, i.e. as the set of all the points in X, which are limiting points for Ω, but are not inner points of Ω. We always assume that $\mu(\partial\Omega) = 0$.

Let

$$\delta_F(x) = \inf_{y\in F} \varrho(x, y)$$

denote the distance of a point x from the set $F \subseteq X$. By

$$\delta(x) = \delta(x, \partial\Omega) := \inf_{y\in\partial\Omega} \varrho(x, y)$$

we denote the distance of x to the boundary.

We say that the measure μ satisfies *the growth condition* equivalently called the upper Ahlfors N-regular, if

$$\mu B(x, r) \le cr^N, \tag{2.2}$$

where $N > 0$ and $c > 0$ does not depend on x and r.

In this paper we do not assume the measure μ to be doubling, but base ourselves on the growth condition (2.2).