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Asymptotic behaviour of the doubly nonlinear diffusion equation $u_t = \Delta_p u^m$ on bounded domains

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ABSTRACT

We study the homogeneous Dirichlet problem for the doubly nonlinear diffusion equation $u_t = \Delta_p u^m$, where $p > 1$, $m > 0$, posed in a bounded domain in $\mathbb{R}^N$ with homogeneous boundary conditions and with non-negative and integrable initial data. In this paper we consider the degenerate case $m(p-1) > 1$ and the quasilinear case $m(p-1) = 1$. In the first case we establish the large-time behaviour by proving the uniform convergence to a unique asymptotic profile and we also give rates for this convergence. The difference in the second case is that the asymptotic profile is unique up to a constant factor that we determine.

1. Introduction

We are interested in describing the behaviour of non-negative solutions of the homogeneous Dirichlet problem for the doubly nonlinear equation (DNLE) for large times. To be precise, we consider the following initial and boundary value problem

$$\begin{cases} u_t(t,x) = \Delta_p u^m(t,x) & \text{for } t > 0 \text{ and } x \in \Omega, \\ u(0,x) = u_0(x) & \text{for } x \in \Omega, \\ u(t,x) = 0 & \text{for } t > 0 \text{ and } x \in \partial\Omega. \end{cases} \tag{1.1}$$

for $m > 0, p > 1$. The problem is posed in a bounded domain $\Omega \in \mathbb{R}^N$ with initial data $u_0 \geq 0,\ u_0 \in L^1(\Omega)$ so that the solution $u(x,t) \geq 0$ too. The p-Laplacian operator is well-known to be defined as $\Delta_p w := \text{div}(|\nabla w|^{p-2}\nabla w)$. We study the large time asymptotic behaviour of solutions to Problem (1.1) in the "degenerate case" $m(p-1) > 1$, also known as the slow diffusion case, and in the "quasilinear case" $m(p-1) = 1$.

Let us first make some comments concerning the range $m(p-1) > 1$. When $p = 2$ we recover the porous medium equation (PME) $u_t = \Delta u^m$ with $m > 1$ while, when $m = 1$, we recover the degenerate p-Laplacian equation (PLE) $u_t = \Delta_p u$

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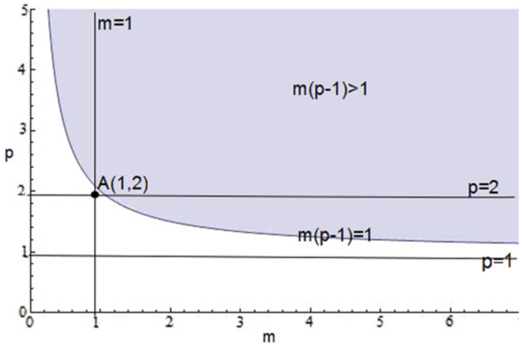

Fig. 1. Ranges of parameters m and p.

with $p > 2$, both are well known equations in the literature. Notice that in this paper we only require $m(p-1) > 1$, that also includes cases where either $m \le 1$ or $p \le 2$. The PLE and the PME, as prototypes for degenerate diffusion, enjoy many common properties, such as finite speed of propagation and the existence of some special (self-similar) solutions, which play an important role in describing the asymptotic behaviour for general initial data. In this paper we complete the panorama by analysing in detail the large-time properties of the degenerate DNLE, which combines the difficulties of both equations and offers some new challenges.

The quasilinear case $m(p-1) = 1$ is also interesting to study since it inherits some common features of the Heat Equation, $u_t = \Delta u$ (which can be recovered when $m = 1$ and $p = 2$): this equation is invariant under scalar multiplication, and it is known that a general solution converges after rescaling to one of the (stationary) solutions of the eigenvalue problem for the p-Laplacian operator. However, when $(m, p) \neq (1, 2)$ differences appear at the level of regularity and qualitative behaviour. While solutions of the HE are C^∞ smooth, solutions of the DNLE have limited regularity due to the degenerate (singular) parabolic character of the equations at the level $u = 0$ (see Fig. 1).

The remaining "fast diffusion case" $m(p-1) < 1$ has quite different properties and deserves a separate study. Indeed, we deal in this case with singular diffusions, and new phenomena appear such as extinction in finite time, or lack of uniqueness of the asymptotic profile. All this gives a different flavour to the analysis of the asymptotic behaviour.

As references for the previous theory for the DNLE we mention [1] for the degenerate and quasilinear cases and [2] for the singular case. We mention also that the asymptotic behaviour of the Cauchy problem on $\mathbb{R}^N$ has been studied in [3]. Many of our results are new even in the p-Laplacian case $m = 1$, $p > 2$. We also remark that most of the techniques needed to prove existence, uniqueness and other basic properties of the parabolic DNLE flow, can be taken from the books [4,5] for the PME, and [6] for the PLE. We also refer the reader to [7,8] for a complete asymptotic analysis of the Dirichlet problem on bounded domains, for the PME when $m > 1$.

Presentation of the main results. The purpose of this work is to analyse completely the asymptotic behaviour of the DNLE on Euclidean bounded domains. For convenience we assume that the boundary $\partial\Omega$ is $C^{2,\alpha}$ smooth. Since the cases $m(p-1) > 1$ and $m(p-1) = 1$ involve different techniques, we will present them separately.

Ia. *The degenerate case* $m(p-1) > 1$. This work generalizes the asymptotic analysis carried out in the above mentioned papers [7,8]. The outline of the theory is similar but the double nonlinearity asks for a number of interesting techniques. Throughout the study we will fix the notation $\mu = 1/(m(p-1)-1) > 0$, since this quantity will appear frequently.

The asymptotic behaviour is better understood via the well-known method of rescaling and time transformation; let us introduce

$$v(\tau, x) = t^\mu u(t, x), \quad t = e^\tau. \tag{1.2}$$

In this way, Problem (1.1) is transformed into

$$\begin{cases} v_\tau(\tau, x) = \Delta_p v^m(\tau, x) + \mu\, v(\tau, x), & \text{for } \tau \in \mathbb{R} \text{ and } x \in \Omega, \\ v(\tau, x) = 0, & \text{for } \tau \in \mathbb{R} \text{ and } x \in \partial\Omega, \\ v(0, x) = v_0 & \text{for } x \in \Omega. \end{cases} \tag{1.3}$$

In Section 2 we prove Theorem 2.1, which shows uniform convergence of the rescaled solution $v(\tau, x)$ to its unique asymptotic profile $f(x)$, as $\tau \to +\infty$. The stationary profile f can be characterized as the positive solution to the corresponding stationary problem

$$\Delta_p f^m + \mu f = 0, \quad \text{in } \Omega,\ f = 0 \text{ on } \partial\Omega.$$

The result of this theorem is not surprising, but it does not appear explicitly in the literature and it is needed to prove the next results. The techniques used in this step follow the work [8] for the PME.

In Section 3 we prove sharp rates of convergence of $v(\cdot, \tau) \to f$ as $\tau \to \infty$; this represents the first important result of this paper.