Nonlinear Analysis 77 (2013) 45–58

Contents lists available at SciVerse ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Asymptotic behavior of increasing solutions to a system of n nonlinear differential equations

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ARTICLE INFO

Article history:
Received 13 July 2012
Accepted 30 August 2012
Communicated by Enzo Mitidieri

MSC:
34C11
34C41
34Exx
35Jxx

Keywords:
Increasing solution
Asymptotic formula
Quasilinear system
Emden–Fowler system
Elliptic system
Regular variation

ABSTRACT

We consider the system $x_i' = a_i(t)|x_{i+1}|^{\alpha_i}\operatorname{sgn} x_{i+1}$, $i = 1, \dots, n$, $n \ge 2$, where a_i, $i = 1, \dots, n$, are positive continuous functions on $[a, \infty)$, $\alpha_i \in (0, \infty)$, $i = 1, \dots, n$, with $\alpha_1 \cdots \alpha_n < 1$, and x_{n+1} means x_1. We analyze the asymptotic behavior of the solutions to this system whose components are eventually positive increasing. In particular, we derive exact asymptotic formulas for the extreme case, where all the solution components tend to infinity (the so-called strongly increasing solutions). We offer two approaches: one is based on the asymptotic equivalence theorem, and the other utilizes the theory of regular variation. The above-mentioned system includes, as special cases, two-term nonlinear scalar differential equations of arbitrary order n and systems of $n/2$ second-order nonlinear equations (provided n is even), which are related to elliptic partial differential systems. Applications to these objects are presented and a comparison with existing results is made. It turns out that some of our results yield new information even in the simplest case, a second-order Emden–Fowler differential equation.

1. Introduction

Consider the nonlinear differential system (of Emden–Fowler type)

$$\begin{cases} x_1' = a_1(t)\Phi_{\alpha_1}(x_2), \\ x_2' = a_2(t)\Phi_{\alpha_2}(x_3), \\ \vdots \\ x_{n-1}' = a_{n-1}(t)\Phi_{\alpha_{n-1}}(x_n), \\ x_n' = a_n(t)\Phi_{\alpha_n}(x_1), \end{cases} \tag{1}$$

$n \in \mathbb{N}$, $n \ge 2$, where $\Phi_\zeta(u) = |u|^\zeta \operatorname{sgn} u$, α_i, $i = 1, \dots, n$, are positive constants, and a_i, $i = 1, \dots, n$, are positive continuous functions defined on $[a, \infty)$, $a \ge 0$. Throughout, we assume that

$$\alpha_1 \cdots \alpha_n < 1; \tag{2}$$

system (1) satisfying this condition will be called *subhomogeneous* (an alternative terminology is *sub-half-linear*). The opposite (strict) inequality is called *superhomogeneity* (or *super-half-linearity*).

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doi:10.1016/j.na.2012.08.019

We study positive solutions of (1), i.e. solutions whose components are eventually positive. Because of the sign conditions on the coefficients, all the components are then eventually increasing. While the asymptotic behavior (as $t \to \infty$) of the solutions where at least one of the components tends to a finite constant is reasonably clear (indeed, an asymptotic formula can then be easily obtained), this is not the case of the so-called strongly increasing solutions. Strongly increasing solutions are defined as those for which all the components tend to infinity. The purpose of this paper is to describe the asymptotic behavior of these "difficult" solutions in an exact way. We offer two approaches: one uses an asymptotic equivalence theorem, and the other utilizes the theory of regular variation. In particular, we show that, if the coefficients in (1) are regularly varying, then so are the components of strongly increasing solutions, and the exact asymptotic formula is derived. The main results are presented in Section 3 and their proofs can be found in the last section.

System (1) covers many important objects studied in the literature, for example, two-term quasilinear differential equations of (odd as well as even) order n, or systems of second-order nonlinear equations which are closely related to certain partial differential systems. We discuss how our results can be applied to these objects and make a comparison with existing theory. We will see that some of our results are new even in the simplest case, i.e., extensively studied second-order quasilinear (or generalized Emden–Fowler) differential equations. These discussions, along with further observations and directions for a future research, can be found in Section 4.

Note that — in some situations — the above-mentioned scalar nth-order differential equations and/or second-order differential systems viewed as systems of the form (1) enable better understanding of the structure of a solution space. In particular, some "symmetry" properties which are otherwise hard to detect can be revealed. Moreover, many of the considerations can be made also for system (1), in which the coefficients are allowed to attain zero values; we will highlight the relevant parts in the text.

This note is organized as follows. In Section 2, we introduce useful conventions, recall basic information about the theory of regular variation, and discuss elementary properties of positive solutions to (1) along with related topics. The content of all other sections is already described in the previous paragraphs.

2. Preliminaries

Conventions We introduce the following convention, which will be throughout used. By a subscript $k \in \mathbb{N}$ we mean $k = i$, where $i \in \{1, \dots, n\}$ and $k \equiv i \pmod n$. Then, for a subscript k,

$$k = \begin{cases} k & \text{if } k \le n, \\ k - mn & \text{if } k > n, \end{cases} \tag{3}$$

where $m \in \mathbb{N}$ is such that $1 \le k - mn \le n$. With this convention, system (1) takes the form

$$x_i' = a_i(t)\Phi_{\alpha_i}(x_{i+1}), \quad i = 1, \dots, n.$$

Further, we adopt the usual convention $\prod_{j=k}^{k-1} u_j = 1$. Recall that, for eventually positive functions $f, g, f(t) \sim g(t)$ as $t \to \infty$ means that $\lim_{t\to\infty} f(t)/g(t) = 1$ and $f(t) \asymp g(t)$ as $t \to \infty$ means that there exist $c_1, c_2 \in (0, \infty)$ such that $c_1 g(t) \le f(t) \le c_2 g(t)$ for large t.

Regular variation The concept of regularly varying functions plays an important role in our theory. A measurable function $f : [a, \infty) \to (0, \infty)$ is called *regularly varying (at infinity) of index* ϑ if

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \lambda^{\vartheta} \quad \text{for every } \lambda > 0;$$

we write $f \in \mathcal{RV}(\vartheta)$. If $\vartheta = 0$, then f is called slowly varying; we write $f \in \mathcal{SV}$. We recommend the monographs [1,2] as very good sources of information on the theory of regular variation. The proofs of the following properties of regularly varying functions, which find applications in our considerations, can also be found there.

- $f \in \mathcal{RV}(\vartheta)$ if and only if
$$f(t) = t^{\vartheta} L(t), \tag{4}$$
where $L \in \mathcal{SV}$.
- If $L_1, \dots, L_n \in \mathcal{SV}$, $n \in \mathbb{N}$, and $R(x_1, \dots, x_n)$ is a rational function with positive coefficients, then $R(L_1, \dots, L_n) \in \mathcal{SV}$. In particular,
$$f_1 f_2 \in \mathcal{RV}(\vartheta_1 + \vartheta_2) \quad \text{and} \quad f_1^{\gamma} \in \mathcal{RV}(\gamma\vartheta_1) \tag{5}$$
for $f_i \in \mathcal{RV}(\vartheta_i)$, $i = 1, 2$, and $\gamma \in \mathbb{R}$. Moreover, $L_1 \circ L_2 \in \mathcal{SV}$ provided $L_2(t) \to \infty$ as $t \to \infty$.
- If $L \in \mathcal{SV}$ and $\vartheta > 0$, then $t^{\vartheta} L(t) \to \infty$, $t^{-\vartheta} L(t) \to 0$ as $t \to \infty$.