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Global existence and blow-up of solutions for a general class of doubly dispersive nonlocal nonlinear wave equations

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A B S T R A C T

This study deals with the analysis of the Cauchy problem of a general class of nonlocal nonlinear equations modeling the bi-directional propagation of dispersive waves in various contexts. The nonlocal nature of the problem is reflected by two different elliptic pseudodifferential operators acting on linear and nonlinear functions of the dependent variable, respectively. The well-known doubly dispersive nonlinear wave equation that incorporates two types of dispersive effects originated from two different dispersion operators falls into the category studied here. The class of nonlocal nonlinear wave equations also covers a variety of well-known wave equations such as various forms of the Boussinesq equation. Local existence of solutions of the Cauchy problem with initial data in suitable Sobolev spaces is proven and the conditions for global existence and finite-time blow-up of solutions are established.

1. Introduction

In this study we mainly establish local existence, global existence and blow-up results for solutions of the Cauchy problem

$$u_{tt} - Lu_{xx} = B(g(u))_{xx}, \quad x \in \mathbb{R},\ t > 0, \tag{1.1}$$
$$u(x,0) = \varphi(x), \qquad u_t(x,0) = \psi(x), \tag{1.2}$$

where g is a sufficiently smooth nonlinear function, L and B are linear pseudodifferential operators defined by

$$\mathcal{F}\,(Lv)\,(\xi) = l(\xi)\mathcal{F}(v)(\xi), \qquad \mathcal{F}\,(Bv)\,(\xi) = b(\xi)\mathcal{F}(v)(\xi).$$

Here $\mathcal{F}$ denotes the Fourier transform with respect to variable x and $l(\xi)$ and $b(\xi)$ are the symbols of L and B, respectively. We assume that L is an elliptic coercive operator of order ρ with $\rho \ge 0$ while B is an elliptic positive operator of order $-r$ with $r \ge 0$. In terms of $l(\xi)$ and $b(\xi)$, this means that there are positive constants c_1, c_2 and c_3 so that for all $\xi \in \mathbb{R}$,

$$c_1^2(1+\xi^2)^{\rho/2} \le l(\xi) \le c_2^2(1+\xi^2)^{\rho/2}, \tag{1.3}$$
$$0 < b(\xi) \le c_3^2(1+\xi^2)^{-r/2}. \tag{1.4}$$

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We emphasize the fact that, for non-polynomial function $l(\xi)$ or nonzero $b(\xi)$, the equation under investigation is of nonlocal type. While the operator L is associated with the regularization resulting from the linear dispersion, the operator B is associated with the regularization resulting from the smoothing of the nonlinear term. In order to reflect more clearly the double nature of the dispersive effects, it is convenient to rewrite (1.1) in a slightly different form. Taking $B = (I + M)^{-1}$ where I is the identity operator and M is an elliptic positive pseudodifferential operator of order $r > 0$, we rewrite (1.1) in the form

$$u_{tt} - \tilde{L}u_{xx} + Mu_{tt} = (g(u))_{xx} \tag{1.5}$$

with $\tilde{L} = (I + M)L$. The second and third terms on the left-hand side of this equation represent two sources of dispersive effects. The relation $\xi \mapsto \omega^2(\xi) = \xi^2\tilde{l}(\xi)/(1 + m(\xi))$ where $\tilde{l}(\xi)$ and $m(\xi)$ are the symbols of $\tilde{L}$ and M, respectively, will be referred to as the linear dispersion relation for (1.5). Since the symbols of $\tilde{L}$ and M will appear in the numerator and denominator, respectively, of the linear dispersion relation, we informally describe the two dispersive effects as "numerator-based" dispersive effect and a "denominator-based" dispersive effect to emphasize the double nature of dispersion.

Even though our main interest lies primarily in understanding the role of pseudodifferential operators L, B, it is worth noting that when $l(\xi)$ is a polynomial, L becomes a differential operator and similarly that, when $b(\xi)$ equals the reciprocal of a polynomial, B becomes the Green function of the corresponding differential operator. In the polynomial case, the equation under investigation (that is, (1.1) or (1.5)) turns out to be some well-known nonlinear wave equations for suitable choices of the operators $\tilde{L}$ and M. For instance, we may note that, with the substitution $\tilde{L} = 1 - \partial_x^2$ and $M = -\partial_x^2$, (1.5) reduces to the so-called double dispersion equation

$$u_{tt} - u_{xx} - u_{xxtt} + u_{xxxx} = (g(u))_{xx}. \tag{1.6}$$

This equation is the most familiar example or special case of (1.1) and was derived in many different contexts (see, for instance, [1,2] where it describes the propagation of longitudinal strain waves in a nonlinearly elastic rod). Thus, (1.5) might be referred to as a natural generalization of the double dispersion equation through the nonlocal operators L and B.

We also point out that (1.5) reduces to the Boussinesq equation

$$u_{tt} - u_{xx} + u_{xxxx} = (g(u))_{xx} \tag{1.7}$$

with the substitution $\tilde{L} = 1 - \partial_x^2$ and $M = 0$ (the zero operator), while it becomes the improved (or regularized) Boussinesq equation

$$u_{tt} - u_{xx} - u_{xxtt} = (g(u))_{xx} \tag{1.8}$$

with the substitution $\tilde{L} = I$ and $M = -\partial_x^2$ [3,4]. Also, assuming $L = 0$ and considering the operator B as a convolution

$$(Bv)(x) = (\beta * v)(x) = \int \beta(x - y)v(y)dy$$

with the kernel $\beta(x) = \mathcal{F}^{-1}(b(\xi))$ where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform, we observe that (1.1) reduces to

$$u_{tt} = \left(\int \beta(x - y)g(u(y, t))dy\right)_{xx}. \tag{1.9}$$

This equation was derived in [5] to model the propagation of strain waves in a one-dimensional, homogeneous, nonlinearly and nonlocally elastic infinite medium (see [6,7] for its coupled form and two-dimensional form, respectively). Our inspiration for the present study comes essentially from (1.9) modeling an integral-type nonlocality of elastic materials. In the present study we add to (1.9) the other type of nonlocality, originating from the inclusion of linear higher order gradients, and focus on how the qualitative results obtained for (1.9) in [5] carry over to (1.1).

There is quite an extensive literature on the well-posedness of the Cauchy problem for the Boussinesq equation (1.7) (see e.g., [8–11]), for the improved Boussinesq equation (1.8) and its higher order generalizations (see e.g., [12–17]) and for the double dispersion equation (1.6) (see e.g., [18]). In [5] consideration was given to the well-posedness of the Cauchy problem for the nonlocal equation (1.9). The question that naturally arises is under which conditions the Cauchy problem (1.1)–(1.2) is well-posed and this is the subject of the present study.

The paper is organized as follows. To simplify the presentation, through Sections 2–5, the special case where B is the identity operator will be treated and the modifications that would be needed for the general case will be given in Section 6. That is, in Sections 2–5 the Cauchy problem for the equation

$$u_{tt} - Lu_{xx} = (g(u))_{xx} \tag{1.10}$$

is only considered; while the Cauchy problem (1.1)–(1.2) is considered in Section 6. In Section 2, the required a priori estimates are established for the linearized version of the Cauchy problem. In Section 3, the local existence and uniqueness for the nonlinear Cauchy problem is proven using the contraction mapping principle. The main theorems stating the global existence and uniqueness of the solution are demonstrated in Section 4. The blow-up criterion is presented in Section 5. Finally, in Section 6, the global existence and blow-up results obtained through Sections 2–5 are extended to the Cauchy problem (1.1)–(1.2).