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Renormings failing to have asymptotically isometric copies of ℓ_1 or c_0

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ABSTRACT

It is proved that every equivalent norm in ℓ_1 is the starting point of a ray of norms which fail to have an asymptotically isometric copy of ℓ_1. In the case where X is separable, it is proved that the set of all equivalent norms in X which fail to have an asymptotically isometric copy of ℓ_1 is residual. Similar results are obtained for the case of c_0. In particular, we deduce that every separable Banach space has an equivalent norm for which it fails to contain an asymptotically isometric copy of ℓ_1 and c_0.

1. Introduction

Let $(X, \|\cdot\|)$ be a Banach space and C a convex closed bounded subset of X. A mapping $T : C \to C$ is called *nonexpansive* if for any $x, y \in C$ we have $\|Tx - Ty\| \leq \|x - y\|$. It is said that a Banach space X has the fixed point property (FPP) if every nonexpansive mapping defined from a closed convex bounded subset into itself has a fixed point. The geometry of the Banach space plays a fundamental role to ensure the FPP. For instance, every Hilbert space and every uniformly convex Banach space and more generally, every reflexive Banach space with normal structure verifies the FPP (see the monographs [1] or [2] and the references therein for a broad exposition in metric fixed point theory). It is well-known that the sequence of Banach spaces ℓ_1 and c_0 fails to have the FPP. However, it turns out to be a difficult problem to know if a Banach space in general fails or not this property. Lennard, Dowling and Turett showed that a Banach space X fails to have the FPP whenever X contains an asymptotically isometric copy of either ℓ_1 or c_0 [3,4]. Using the method of finding such copies of ℓ_1 and c_0, many other nonreflexive Banach spaces were discovered to fail the FPP (see for instance Chapter 9 in [2]). However, despite Jame's distortion theorem, which asserts that every renorming of ℓ_1 or c_0 contains an almost isometric copy of ℓ_1 or c_0 respectively, it is also known that there exist some other equivalent norms in ℓ_1 and c_0 which contain no asymptotically isometric copy of ℓ_1 or c_0 respectively.

In this paper we study density and generic properties of the set of all equivalent norms in a Banach space which contains an asymptotically isometric copy of either ℓ_1 or c_0. It is organized as follows: in Section 2 we introduce the necessary background and the notation which will be used throughout the paper. In Section 3 we prove that every equivalent norm

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in ℓ_1 is the starting point of a ray of equivalent norms which fail to have an asymptotically isometric copy of ℓ_1. This kind of linear structure will let us deduce that the previous set is dense in the set of all equivalent norms in ℓ_1. We will improve this density result for separable Banach spaces by using residual methods. We will check that our results do not hold in general in the nonseparable case. We will conclude the paper with similar achievements for the case of c_0. As a consequence, we will show that every separable Banach space has an equivalent norm which fails to have both an asymptotically isometric copy of ℓ_1 and c_0.

2. Preliminaries

The fact that the Banach spaces ℓ_1 and c_0 fail to have the FPP can be shown by using that the right shift operator is nonexpansive for the usual norm in ℓ_1 and c_0. Nevertheless, proving if a general Banach space fails or not the FPP is a difficult problem.

We recall some basic definitions for the purposes of this paper.

Definition 2.1. A Banach space X is said to have an asymptotically isometric copy of ℓ_1 if there are a null sequence (ε_n) in $(0, 1)$ and a sequence (x_n) in X so that

$$\sum_{n=1}^{\infty}(1-\varepsilon_n)|t_n| \le \left\|\sum_{n=1}^{\infty} t_n x_n\right\| \le \sum_{n=1}^{\infty}|t_n|,$$

for all $(t_n) \in \ell_1$. In this case we say that X contains a ℓ_1-a.i. copy.

Definition 2.2. A Banach space X is said to have an asymptotically isometric copy of c_0 if there are a null sequence (ε_n) in $(0, 1)$ and a sequence (x_n) in X so that

$$\sup_{n\in\mathbb{N}}(1-\varepsilon_n)|t_n| \le \left\|\sum_{n=1}^{\infty} t_n x_n\right\| \le \sup_{n\in\mathbb{N}}|t_n|,$$

for all $(t_n) \in c_0$. In this case we say that X contains a c_0-a.i. copy.

Pelczynski [5] proved that a Banach space contains an isomorphic copy of ℓ_1 if and only if its dual contains an isomorphic copy of $L_1[0, 1]$. The concept of asymptotically isometric copy of ℓ_1 was defined by Hagler in his thesis. In fact, Hagler generalized the previous statement by proving that a Banach space contains an asymptotically isometric copy of ℓ_1 if and only if its dual contains an isometric copy of $L_1[0, 1]$. This result was later published in [6]. Afterwards, Dowling and Lennard and Turret [4] defined, in an analogous way, the concept of asymptotically isometric copy of c_0 and connected the existence of one of such copies with the failure of the FPP. Namely, they proved that if a Banach space contains a sequence generating an asymptotically isometric copy of either ℓ_1 or c_0 then one can 'embed' the fixed point free nonexpansive mapping in ℓ_1 or c_0 respectively into the Banach space X and show that X fails to have the FPP [3,7]. This method is used to prove that every nonreflexive subspace of $L_1[0, 1]$ or $K(H)$ fails to have the FPP (see [8] and also Chapter 9 in [2]).

However, there exist equivalent norms on ℓ_1 and c_0 which contain no ℓ_1-a.i. copy or c_0-a.i. copy respectively. In fact, in [7] it is shown that the following renorming of ℓ_1 fails to have a ℓ_1-a.i. copy: let (γ_n) be a nondecreasing sequence in $(0, 1)$ which converges to 1 and consider the equivalent norm on ℓ_1 given by

$$|||t||| = \sup_n \gamma_n \sum_{k=n}^{\infty}|t_k|, \quad \text{for all } t = (t_n) \in \ell_1.$$

Then $(\ell_1, |||\cdot|||)$ fails to have an asymptotically isometric copy of ℓ_1.

Later, Lin [9] proved that if $\gamma_n = \frac{8^n}{1+8^n}$, then $(\ell_1, |||\cdot|||)$ does have the FPP, showing that FPP does not imply reflexivity, contrary to what had been conjectured for a long time. Since then, Lin's renorming has been extended by several authors to give new nonreflexive Banach spaces with the FPP (see [10,11]). In fact, it can be proved (in a more general framework) that $(\ell_1, |||\cdot|||)$ has the FPP, where (γ_n) is any nondecreasing sequence in $(0, 1)$ tending to 1 [11]. Notice also that $\gamma_1\|x\|_1 \le |||x||| \le \|x\|_1$ for all $x \in \ell_1$ which means that we can find equivalent norms on ℓ_1 as close as we want to $\|\cdot\|_1$ without asymptotically isometric copies of ℓ_1 and satisfying the FPP.

We introduce the following notation: given $(X, \|\cdot\|)$ a Banach space we define by $\mathcal{P}(X)$ the set of all equivalent norms on X endowed with the metric:

$$\rho(p, q) = \sup\{|p(x) - q(x)| : \|x\| \le 1\}, \quad \text{if } p, q \in \mathcal{P}(X).$$

Notice that $\mathcal{P}(X)$ is an open subset of the complete metric space $(\mathcal{Q}, \rho)$ of all continuous seminorms on $(X, \|\cdot\|)$ with the metric ρ defined as above. Thus $(\mathcal{P}(X), \rho)$ is a Baire space. Set

$$\mathcal{P}_{\text{FPP}}(X) := \{p \in X : (X, p) \text{ has the FPP}\},$$

$$\mathcal{P}_1(X) := \{p \in \mathcal{P}(X) : (X, p) \text{ fails to have an } \ell_1\text{-a.i. copy}\},$$