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Study of a 3*D*-Ginzburg–Landau functional with a discontinuous pinning term

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ABSTRACT

In a convex domain $\Omega \subset \mathbb{R}^3$, we consider the minimization of a 3*D*-Ginzburg–Landau type energy $E_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{2\varepsilon^2}(a^2 - |u|^2)^2$ with a discontinuous pinning term a among $H^1(\Omega, \mathbb{C})$-maps subject to a Dirichlet boundary condition $g \in H^{1/2}(\partial\Omega, \mathbb{S}^1)$. The pinning term $a : \mathbb{R}^3 \to \mathbb{R}_+^*$ takes a constant value $b \in (0, 1)$ in ω, an inner strictly convex subdomain of Ω, and 1 outside ω. We prove energy estimates with various error terms depending on assumptions on Ω, ω and g. In some special cases, we identify the vorticity defects via the concentration of the energy. Under hypotheses on the singularities of g (the singularities are polarized and quantified by their degrees which are ± 1), vorticity defects are geodesics (computed w.r.t. a geodesic metric d_{a^2} depending only on a) joining two paired singularities of $g p_i \& n_{\sigma(i)}$ where σ is a minimal connection (computed w.r.t. a metric d_{a^2}) of the singularities of g and $p_1, \dots, p_k$ are the positive (resp. $n_1, \dots, n_k$ are the negative) singularities.

1. Introduction

In a convex domain $\Omega \subset \mathbb{R}^3$, we consider the minimization of a 3D-Ginzburg–Landau type energy with a discontinuous pinning term among $H^1(\Omega, \mathbb{C})$-maps subject to a Dirichlet boundary condition $g \in H^{1/2}(\partial\Omega, \mathbb{S}^1)$. The pinning term $a : \mathbb{R}^3 \to \mathbb{R}_+^*$ takes a constant value $b \in (0, 1)$ in ω, an inner strictly convex subdomain of Ω, and 1 outside ω. The strict convexity of ω is not necessary but it allows to make a simpler description of the techniques used in this article.

Our Ginzburg–Landau type energy is

$$E_\varepsilon(u) = \frac{1}{2}\int_\Omega \left\{ |\nabla u(x)|^2 + \frac{1}{2\varepsilon^2}\left[a(x)^2 - |u(x)|^2\right]^2 \right\} dx. \tag{1}$$

In (1), $u \in H_g^1 := \{u \in H^1(\Omega, \mathbb{C}) \mid \mathrm{tr}_{\partial\Omega} u = g\}$.

We are interested in studying the *vorticity defects* of minimizers of E_ε in H_g^1 via energetic estimates. In this article, letting u_ε be such a minimizer, we aim in describing the set $\{|u_\varepsilon| \le b/2\}$ (this set is the vorticity defects). In the asymptotic $\varepsilon \to 0$ we expect that, at least for special g's, the set $\{|u_\varepsilon| \le b/2\}$ takes the form of a union of thin wires whose endpoints are in $\partial\Omega$; under this form the vorticity defects are called *vorticity lines*. We also expect that a concentration of the energy occurs around this set.

Because the pinning term is discontinuous, an energetical noise appears in a small neighborhood of the discontinuity set of a (this set is $\partial\omega$).

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In order to study the minimization problem of E_ε in H^1_g we first consider an auxiliary minimization problem. Following [1], we let U_ε be *the* unique minimizer of E_ε in $H^1_1 := \{u \in H^1(\Omega, \mathbb{C}) \mid \operatorname{tr}_{\partial\Omega} u \equiv 1\}$. The solution U_ε plays an important role in the study. It allows to consider a decoupling of E_ε (see Section 2). If $v \in H^1(\Omega, \mathbb{C})$ and $|v| \equiv 1$ on $\partial\Omega$, then [1]

$$E_\varepsilon(U_\varepsilon v) = E_\varepsilon(U_\varepsilon) + F_\varepsilon(v), \quad \text{where } F_\varepsilon(v) = \frac{1}{2}\int_\Omega \left\{U_\varepsilon^2 |\nabla v|^2 + \frac{U_\varepsilon^4}{2\varepsilon^2}(1-|v|^2)^2\right\}.$$

Consequently the study of minimizers of E_ε in H^1_g is related to the study of minimizers of F_ε in H^1_g.

Our techniques are directly inspired from those initially developed by Sandier in [2] (whose purpose was to give, in some special situations, a simple proof of the 3D analysis of the Ginzburg–Landau equation, by Lin and Rivière [3]), and by their adaptations in [4].

We prove energy estimates with various error terms depending on our assumptions on Ω and g (see Theorems 2–4 in Section 4). In some special cases, we identify the vorticity lines via the concentration of the energy. At the end of this section, we will present a strategy which could lead to the localization of the vorticity lines.

The results we present are a first step towards a more precise description of the vorticity defects and of the asymptotic of minimizers.

Before stating our own results, we start by recalling the asymptotic expansion of the energy in the standard 3D-Ginzburg–Landau model (when $a \equiv 1$).

For $g \in H^{1/2}(\partial\Omega, \mathbb{S}^1)$, if we let

$$E^0_\varepsilon(u) = \frac{1}{2}\int_\Omega \left\{|\nabla u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2\right\},$$

then we have

$$\inf_{H^1_g} E^0_\varepsilon = C(g)|\ln\varepsilon| + o(|\ln\varepsilon|). \tag{2}$$

Moreover, $\frac{C(g)}{\pi}$ is given by the length of a minimal connection connecting the singularities of g (in the spirit of Brezis et al. [5]). (see [3,6,2,4]).

For special g's and for a convex domain Ω, (2) was obtained by Lin and Rivière [3] (see also [6]) and Sandier [2]. The case of a general data $g \in H^{1/2}(\partial\Omega, \mathbb{S}^1)$ and a simply connected Ω is due to Bourgain et al. [4].

The above articles are the main references in this work. One of our main results is the analog of (2) for the minimization of F_ε (Theorem 2). This result is first proved when g is in a dense set $\mathcal{H} \subset H^{1/2}(\partial\Omega, \mathbb{S}^1)$ and then extended by density. The upper bound is obtained directly using the techniques developed in [2,4]. The lower bound needs an adaptation in the argument of Sandier [2]. The main ingredient used to obtain a lower bound in [2] is the existence of a "*structure function*" adapted to the singularities of g (see Section 7). In the spirit of [2], we prove, under suitable assumptions on Ω, ω and g, the existence of structure functions adapted to our situation (see Propositions 4, 6, 7 and 10).

In our situation ($a = b$ in ω and $a = 1$ in $\mathbb{R}^3 \setminus \omega$), we have an analog of (2) for $\inf_{H^1_g} F_\varepsilon$ replacing $C(g)$ by $C(g, a)$. When g admits a finite number of singularities, the constant $C(g, a)/\pi$ is the length of a minimal connection between the singularities of g (see Section 3 for precise definitions). This minimal connection is computed w.r.t. a metric d_{a^2} depending only on a (see (11)). (This generalizes the case of the standard potential $(1-|u|^2)^2$, where the distance is the euclidean one.)

When g has a finite number of singularities, one may prove a concentration of the energy along the vorticity lines (see Theorems 3 and 4). As in [3,2], we obtain, after normalization, that the energy of minimizer is uniform along the vorticity lines (See Theorem 3). These vorticity lines are identified: they are geodesic segments associated to d_{a^2}.

In order to sum up our main results we state a theorem in a simplified form. This theorem is a direct consequence of Theorems 2–4 stated in Section 4.

Theorem 1. *Let $g \in H^{1/2}(\partial\Omega, \mathbb{S}^1)$ then we have*

- $\inf_{H^1_g} E_\varepsilon = E_\varepsilon(U_\varepsilon) + C(g, a)|\ln\varepsilon| + o(|\ln\varepsilon|)$ *where $E_\varepsilon(U_\varepsilon) \sim \varepsilon^{-1}$ and $C(g, a)$ depends only on the singularities of g and on a (it is the length of a minimal connection of the singularities of g computed w.r.t. the distance d_{a^2}).*
- *Let g be a prepared boundary condition with a finite number of singularities of degree ± 1 ($g \in \mathcal{H}$, $\mathcal{H}$ defined in (7)) and let $p_1, \ldots, p_k$ (resp. $n_1, \ldots, n_k$) be the positive (resp. negative) singularities of g. Let Γ be the geodesic link of the singularities (we assume that Γ is unique), i.e., Γ is a union of k geodesic curves joining p_i with $n_{\sigma(i)}$ where σ is a permutation of $\{1, \ldots, k\}$ s.t. the total d_{a^2}-length of the curves is minimal.*

  Letting v_ε be a minimizer of F_ε in H^1_g (i.e. $U_\varepsilon v_\varepsilon$ minimizes E_ε) we have

$$\frac{\frac{U_\varepsilon^2}{2}|\nabla v_\varepsilon|^2 + \frac{U_\varepsilon^4}{4\varepsilon^2}(1-|v_\varepsilon|^2)^2}{|\ln\varepsilon|}\,\mathcal{H}^3 \quad \text{weakly converges in } \Omega \text{ in the sense of the measures to } \pi a^2 \mathcal{H}^1_{|\Gamma}.$$

Here $\mathcal{H}^3$ is the 3-dimensional Hausdorff measure and $\mathcal{H}^1_{|\Gamma}$ is the one dimensional Hausdorff measure on Γ.