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Classical solutions of hyperbolic IBVPs with state dependent delays on a cylindrical domain

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ABSTRACT

We consider the initial boundary value problem for a nonlinear partial functional differential equation of the first order

$$\partial_t z(t,x) = f(t,x,V(z;t,x),\partial_x z(t,x)),$$

where V is a nonlinear operator of Volterra type, mapping bounded subsets of the space of Lipschitz-continuously differentiable functions, into bounded subsets of the space of Lipschitz continuous functions with Lipschitz continuous spatial partial derivatives. Using the method of bicharacteristics and successive approximations, we prove the local existence, uniqueness and continuous dependence on data of classical solutions of the problem. This approach covers equations of the form

$$\partial_t z(t,x) = f(t,x,z_{\alpha(t,x,z_{(t,x)})},\partial_x z(t,x)),$$

where $(t,x) \mapsto z_{(t,x)}$ is the (multidimensional) Hale operator, and all the components of α may depend on $(t,x,z_{(t,x)})$. More specifically, problems with deviating arguments and integro-differential equations are included.

1. Introduction

We formulate the functional differential problem. Let $a > 0$, $b_0 \geq 0$, and the domain $\Omega \subset \mathbb{R}^n$ with C^1 boundary, be given. Write $E_0 = [-b_0, 0] \times \overline{\Omega}$, $E = [0,a] \times \overline{\Omega}$, $\partial_0 E = (0,a] \times \text{bdry}\,\Omega$, where bdry Ω denotes the boundary of Ω. We will write $C^m(\overline{U})$ for the space of functions having, on U, bounded and uniformly continuous partial derivatives up to the order m. The case $m = 0$ has here its usual meaning: $C^0 \equiv C$. Elements of this space are meant to be real-valued; its vector-valued analogue will be denoted by $C^m(\overline{U}, \mathbb{R}^n)$. We write $C^{m\cdot L}(\overline{U})$ and $C^{m\cdot L}(\overline{U}, \mathbb{R}^n)$ for the respective spaces of functions having, on U, bounded and Lipschitz continuous partial derivatives up to the order m. Put $D = [-b_0 - a, 0] \times (\overline{\Omega} - \overline{\Omega})$, where the subtraction $\overline{\Omega} - \overline{\Omega}$ is understood as the set of all possible subtractions of points from $\overline{\Omega}$. We consider the problem

$$\partial_t z(t,x) = f(t,x,V(z;t,x),\partial_x z(t,x)) \quad \text{on } E \tag{1}$$

$$z(t,x) = \varphi(t,x) \quad \text{on } E_0 \cup \partial_0 E, \tag{2}$$

with $V : C^{1\cdot L}(\overline{E^*}) \times E \to C(\overline{D})$, $f : E \times C(\overline{D}) \times \mathbb{R}^n \to \mathbb{R}$. Let $E^* = E_0 \cup E \cup \partial_0 E$.

When it does not lead to misunderstanding, we write $U_t = U \cap ([-\infty, t] \times \mathbb{R}^n)$ for $U \subset \mathbb{R}^{1+n}$ and $t \in [0,a]$. We assume that V is a nonlinear Volterra operator; by the Volterra property we mean that for $z, \bar{z} \in C^{1\cdot L}(\overline{E^*})$ and $(t,x) \in E$,

$$z|_{E_t^*} \equiv \bar{z}|_{E_t^*} \quad \text{implies } V(z;t,x) \equiv V(\bar{z};t,x). \tag{3}$$

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A function $\tilde{z} \in C^1(\overline{E_c^*})$, where $0 < c \le a$, is a classical solution of (1), (2) if it satisfies (1) on E_c and condition (2) holds on $E_0 \cup \partial_0 E_c$.

Note that different models of the functional dependence in partial equations are used in the literature. The first group of results is connected with initial problems for equations

$$\partial_t z(t,x) = G(t,x,z,\partial_x z(t,x)) \tag{4}$$

where the variable z represents the functional argument. This model is suitable for differential functional inequalities generated by initial problems considered on the Haar pyramid. Existence results for (4) can be characterized as follows: theorems have simple assumptions and their proofs are quite natural (see [1,2]). Unfortunately, only a small class of differential functional problems is covered by this theory. There are a lot of papers concerning initial value problems for equations

$$\partial_t z(t,x) = H(t,x,W[z](t,x),\partial_x z(t,x)) \tag{5}$$

where W is an operator of Volterra type and H is defined on the finite-dimensional Euclidean space. The main assumptions in existence theorems for (5) concern the operator W. They are formulated [3,4] in terms of inequalities for norms in some functional spaces.

A new model of a functional dependence is presented in [5,6]. Partial equations have the form

$$\partial_t z(t,x) = F(t,x,z_{(t,x)},\partial_x z(t,x)) \tag{6}$$

where $z_{(t,x)}$ is a functional variable. This model is well-known for ordinary functional differential equations (see, for example, [7–9]). It is also very general since equations with deviating variables, integral differential equations, and equations of forms (4) and (5) can be obtained from (6) by specifying the operator F. In the paper we use the model (6). In existence results, concerning partial differential equations with state dependent delays [10–12], Carathéodory type or semiclassical solutions were considered and the functional variable was

$$z_{(\psi_0(t),\psi'(t,x,z_{(t,x)}))}.$$

We deal in this paper with a slightly wider class of deviating functions, admitting the functional variable of the form

$$z_{(\psi_0(t,x,z_{(t,x)}),\psi'(t,x,z_{(t,x)}))}$$

and we consider classical solutions of the respective problem. Cases of more (or less) complicated deviating functions are also covered by our operator formulation.

Delay systems with state dependent delays occur as models for the dynamics of diseases when the mechanism of infection is such that the infectious dosage received by an individual has to reach a threshold before the resistance of the individual is broken down and as a result the individual becomes infectious. A prototype of such model was proposed in [13].

The aim of this paper is to prove a theorem on the existence and continuous dependence of classical solutions to (1), (2).

2. The domain and function spaces

By a convenient abuse of notation, let $|\cdot|$ stand for the Euclidean norm in $\mathbb{R}^n$. For k, l being arbitrary positive integers, we denote by $M_{k\times l}$ the class of all $k \times l$ matrices with real elements, and we choose the norms in $\mathbb{R}^k$ and $M_{k\times l}$ to be ∞-norms: $\|y\| = \|y\|_\infty = \max_{1\le i\le k}|y|$ and $\|A\| = \|A\|_\infty = \max_{1\le i\le k}\sum_{j=1}^{l}|a_{ij}|$, respectively, where $A = [a_{ij}]_{i=1,\dots,k,j=1,\dots,l}$.

We assume that the domain Ω satisfies a uniform $C^{1.L}$-regularity condition, which is a variant of the one from [14], concerning C^1-regularity. Let Φ be a one-to-one transformation of a domain $U \subset \mathbb{R}^n$ onto a domain $V \subset \mathbb{R}^n$, having inverse $\Psi = \Phi^{-1}$. We say that Φ is 1-*smooth* if all components of Φ and Ψ belong to $C^1(\overline{U})$ and to $C^1(\overline{V})$, respectively. Let Ω_δ denote the set of points in Ω within distance δ of the boundary of Ω:

$$\Omega_\delta = \{x \in \Omega : \text{dist}(x, \text{bdry}\,\Omega) < \delta\}.$$

Assumption H[Ω]. There exist a locally finite open cover $\{U_j\}$ of bdry Ω, and a corresponding sequence $\{\Phi_j\}$ of 1-smooth transformations, with Φ_j taking U_j onto the unit ball $B = \{y \in \mathbb{R}^n : |y| < 1\}$ and having inverse $\Psi_j = \Phi_j^{-1}$, such that we have the following.

(i) For some $\delta > 0$, $\Omega_\delta \subset \bigcup_{j=1}^{\infty}\Psi_j(\{y \in \mathbb{R}^n : |y| < 1/2\})$.

(ii) For each j, $\Phi_j(U_j \cap \Omega) = \{y \in B : y_n > 0\}$.

(iii) There is a finite constant M such that for every j

$$\|\partial_x \Phi_j(x)\| \le M, \quad \text{for } x \in U_j,$$
$$\|\partial_y \Psi_j(y)\| \le M, \quad \text{for } y \in B,$$

that is, the norms of Jacobi matrices for Φ_j and Ψ_j are uniformly bounded by M.

(iv) There is a finite constant L_ϕ such that for any $j, x, \bar{x} \in U_j$, $1 \le i \le n-1$, there is $\|\partial_x \phi_{ji}(x) - \partial_x \phi_{ji}(\bar{x})\| \le L_\phi \|x - \bar{x}\|$.