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Generalized Cauchy type problems for nonlinear fractional differential equations with composite fractional derivative operator

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ABSTRACT

This paper is devoted to proving the existence and uniqueness of solutions to Cauchy type problems for fractional differential equations with composite fractional derivative operator on a finite interval of the real axis in spaces of summable functions. An approach based on the equivalence of the nonlinear Cauchy type problem to a nonlinear Volterra integral equation of the second kind and applying a variant of the Banach's fixed point theorem to prove uniqueness and existence of the solution is presented. The Cauchy type problems for integro-differential equations of Volterra type with composite fractional derivative operator, which contain the generalized Mittag-Leffler function in the kernel, are considered. Using the method of successive approximation, and the Laplace transform method, explicit solutions of the open problem proposed by Srivastava and Tomovski (2009) [11] are established in terms of the multinomial Mittag-Leffler function.

1. Introduction and preliminaries

Applications of fractional calculus require fractional derivatives of different kinds (see [1,2]). Differentiation and integration of functions of fractional order are traditionally defined using Riemann–Liouville (R–L) operators $I_{a+}^{\mu}f$, $D_{a+}^{\mu}f$ as [3–6]

$$(I_{a+}^{\mu}f)(x) = \frac{1}{\Gamma(\mu)}\int_a^x \frac{f(t)}{(x-t)^{1-\mu}}dt, \quad (\mu \in \mathbb{C}, \Re(\mu) > 0) \tag{1.1}$$

$$(D_{a+}^{\mu}f)(x) = \left(\frac{d}{dx}\right)^n (I_{a+}^{n-\mu}f)(x), \quad (\mu \in \mathbb{C}, \Re(\mu) \geq 0,\ n = [\Re(\mu)] + 1) \tag{1.2}$$

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where $a < x$, f is locally integrable, and $\Re$ denotes the real part. The operator (1.1) is defined on the space $L(a, b)$ of Lebesgue measurable functions $f(x)$ on a finite interval $[a, b]$ $(b > a)$ of the real line $\mathbb{R}$:

$$L(a, b) = \left\{ f : \|f\|_1 = \int_a^b |f(x)| dx < \infty \right\}.$$

Let $AC[a, b]$ be the space of real-valued functions $f(x)$ which are absolutely continuous on $[a, b]$. For $n \in \mathbb{N}$, we denote by $AC^n[a, b]$ the space of real-valued functions $f(x)$ which have continuous derivatives up to order $n - 1$ on $[a, b]$ such that $f^{(n-1)}(x) \in AC^n[a, b]$:

$$AC^n[a, b] = \left\{ f : [a, b] \to \mathbb{R} : (D^{n-1}f)(x) \in AC[a, b] \left(D = \frac{d}{dx} \right) \right\}.$$

More recently, in [1,7,8] an infinite family of fractional (R–L) derivatives having the same order were introduced as follows.

Definition 1. The (right-hand side) fractional derivative $D_{a+}^{\mu,\nu}$ of order $0 < \mu < 1$ and type $0 \le \nu \le 1$ with respect to x is defined by

$$(D_{a+}^{\mu,\nu} f)(x) = \left(I_{a+}^{\nu(1-\mu)} \frac{d}{dx} (I_{a+}^{(1-\nu)(1-\mu)} f) \right)(x) \tag{1.3}$$

whenever the right-hand side exists. This generalization gives the classical (R–L) fractional differentiation operator if $\nu = 0$. For $\nu = 1$ it gives the fractional differential operator introduced by Liouville on page 10 in [9] but nowadays often named after Caputo. Several authors (see [10,11]) called (1.3) the Hilfer fractional derivative or composite fractional derivative operator. Applications of $D_{a+}^{\mu,\nu}$ are given in [7,2,12,13].

Recently (Hilfer et al. [14]), this definition for $n - 1 < \mu \le n$, $n \in \mathbb{N}$, $0 \le \nu \le 1$, was rewritten in a more general form:

$$(D_{a+}^{\mu,\nu} f)(x) = \left(I_{a+}^{\nu(n-\mu)} \frac{d^n}{dx^n} (I_{a+}^{(1-\nu)(n-\mu)} f) \right)(x) = (I_{a+}^{\nu(n-\mu)} D_{a+}^{\mu+\nu n-\mu\nu} f)(x) \tag{1.4}$$

The difference between fractional derivatives of different types becomes apparent from Laplace transformation. In [1,7] it is found for $0 < \mu < 1$ that

$$\mathcal{L}[D_{0+}^{\mu,\nu} f(x)](s) = s^{\mu} \mathcal{L}[f(x)](s) - s^{\nu(\mu-1)} (I_{0+}^{(1-\nu)(1-\mu)} f)(0+) \tag{1.5}$$

where $(I_{0+}^{(1-\nu)(1-\mu)} f)(0+)$ is the R–L integral of order $(1 - \nu)(1 - \mu)$ evaluated in the limit as $t \to 0+$, it being understood (as usual) that

$$\mathcal{L}[f(x)](s) = \int_0^{\infty} e^{-sx} f(x) dx, \tag{1.6}$$

provided that the defining integral in (1.6) exists.

Some compositional properties with fractional derivative operator (1.3) and Hardy type inequality with generalized R–L fractional derivative operator were obtained in the recent papers [11,15]. Using the Laplace transform method, several fractional differential equations with constant and variable coefficients, as well as some Volterra differintegral equations in the space of summable functions were also solved [11,15]. An operational calculus of the Mikusinski type was introduced for the generalized R–L fractional derivative operator [14] and it was used to solve the corresponding initial value problem for the general n-term linear fractional differential equation with constant coefficients with generalized R–L fractional derivatives of arbitrary orders and types. The solution of such an equation is expressed in terms of multinomial Mittag-Leffler functions, which will be introduced later, in this section.

The Mittag-Leffler (M-L) functions $E_{\alpha}(z)$ and $E_{\alpha,\beta}(z)$ are defined by the following series:

$$E_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)} \quad (\alpha \in \mathbb{C}, \Re(\alpha) > 0) \tag{1.7}$$

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)} \quad (\alpha, \beta \in \mathbb{C}, \Re(\alpha) > 0). \tag{1.8}$$

By means of the series representation a generalization of the M-L function $E_{\alpha,\beta}^{\gamma}(z)$ of (1.8) was introduced by Prabhakar [16] as follows:

$$E_{\alpha,\beta}^{\gamma}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n z^n}{\Gamma(\alpha n + \beta) n!} \quad (\alpha, \beta, \gamma \in \mathbb{C}, \Re(\alpha) > 0) \tag{1.9}$$