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Sell's conjecture for non-autonomous dynamical systems

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ABSTRACT

This paper is dedicated to the study of the G. Sell's conjecture for general non-autonomous dynamical systems. We give a positive answer for this conjecture and we apply this result to different classes of non-autonomous evolution equations: Ordinary Differential Equations, Functional Differential Equations and Semi-linear Parabolic Equations.

1. Introduction

The aim of this paper is the study the problem of global asymptotic stability of trivial solution for non-autonomous dynamical systems. We study this problem in the framework of general *non-autonomous dynamical systems* (NDS).

Consider a differential equation

$$x' = f(t,x) \quad (f \in C(\mathbb{R} \times W, \mathbb{R}^n)), \tag{1}$$

where $\mathbb{R} := (-\infty, +\infty)$, $\mathbb{R}^n$ is a product space of n copies of $\mathbb{R}$, W is an open subset from $\mathbb{R}^n$ containing the origin (i.e., $0 \in W$), $C(\mathbb{R} \times W, \mathbb{R}^n)$ is the space of all continuous functions $f : \mathbb{R} \times W \mapsto \mathbb{R}^n$ equipped with a compact open topology. This topology is defined, for example [1,2], by the following distance:

$$\rho(f,g) := \sum_{k=1}^{+\infty} \frac{1}{2^k} \frac{\rho_k(f,g)}{1+\rho_k(f,g)},$$

where $\rho_k(f,g) := \max\{|f(t,x) - g(t,x)| : (t,x) \in [-k,k] \times W_k\}$, $\{W_k\}$ is a family of compact subsets from W with the properties: $W_k \subset W_{k+1}$ for all $k \in \mathbb{N}$, $\bigcup_{k=1}^{+\infty} W_k = W$, and $|\cdot|$ is a norm on $\mathbb{R}^n$. Denote by $(C(\mathbb{R} \times W, \mathbb{R}^n), \mathbb{R}, \sigma)$ a *shift dynamical system* [1,2] on the space $C(\mathbb{R} \times W, \mathbb{R}^n)$ (*dynamical system of translations* or *Bebutov's dynamical system*), i.e., $\sigma(\tau, f) := f_\tau$ for all $\tau \in \mathbb{R}$ and $f \in C(\mathbb{R} \times W, \mathbb{R}^n)$, where $f_\tau(t,x) := f(t+\tau, x)$ for all $(t,x) \in \mathbb{R} \times W$.

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Below we will use the following conditions:

(A): for all $(t_0, x_0) \in \mathbb{R}_+ \times W$ Eq. (1) admits a unique solution $x(t; t_0, x_0)$ with initial data (t_0, x_0) and defined on $\mathbb{R}_+ := [0, +\infty)$, i.e., $x(t_0; t_0, x_0) = x_0$;

(B): the right hand side f *is positively compact*, if the set $\Sigma_f^+ := \{f_\tau : \tau \in \mathbb{R}_+\}$ is a relatively compact subset of $C(\mathbb{R} \times W, \mathbb{R}^n)$;

(C): the equation

$$y' = g(t, y), \quad (g \in \Omega_f)$$

is called a *limiting equation* for (1), where Ω_f is the ω-limit set of f with respect to the shift dynamical system $(C(\mathbb{R} \times W, \mathbb{R}^n), \mathbb{R}, \sigma)$, i.e., $\Omega_f := \{g :$ there exists a sequence $\{\tau_k\} \to +\infty$ such that $f_{\tau_k} \to g$ as $k \to \infty\}$;

(D): Eq. (1) (or its hand right side f) *is regular*, if for all $p \in H^+(f)$ the equation

$$y' = p(t, y)$$

admits a unique solution $\varphi(t, x_0, p)$ defined on $\mathbb{R}_+$ with initial condition $\varphi(0, x_0, p) = x_0$ for all $x_0 \in W$, where $H^+(f) := \overline{\{f_\tau : \tau \in \mathbb{R}_+\}}$ and by bar we denote the closure in the space $C(\mathbb{R} \times W, \mathbb{R}^n)$;

(E): Eq. (1) admits a null (trivial) solution, i.e., $f(t, 0) = 0$ for all $t \in \mathbb{R}_+$;

(F): a function f satisfies to local (respectively, global) *Lipschitz condition*, if there exists a function $L : \mathbb{R}_+ \mapsto \mathbb{R}_+$ (respectively, a positive constant L) such that

$$|f(t, x_1) - f(t, x_2)| \le L(r)|x_1 - x_2|$$

(respectively, $|f(t, x_1) - f(t, x_2)| \le L|x_1 - x_2|$) for all $t \in \mathbb{R}_+$ and $x_1, x_2 \in W$ with $|x_1|, |x_2| \le r$ for all $r > 0$ (respectively, for all $x_1, x_2 \in W$).

The trivial solution of Eq. (1) is said to be:

(i) *uniformly stable*, if for all positive number ε there exists a number $\delta = \delta(\varepsilon)$ ($\delta \in (0, \varepsilon)$) such that $|x| < \delta$ implies $|\varphi(t, x, f_\tau)| < \varepsilon$ for all $t, \tau \in \mathbb{R}_+$;

(ii) *uniformly attracting*, if there exists a positive number a

$$\lim_{t \to +\infty} |\varphi(t, x, f_\tau)| = 0$$

uniformly with respect to $|x| \le a$ and $\tau \in \mathbb{R}_+$;

(iii) *uniformly asymptotically stable*, if it is uniformly stable and uniformly attracting.

G. Sell's Conjecture (*[2, Chapter VIII, p. 134]*). Let $f \in C(\mathbb{R} \times W, \mathbb{R}^n)$ be a regular function and f be positively pre-compact. Assume that W contains the origin 0 and $f(t, 0) = 0$ for all $t \in \mathbb{R}_+$. Assume further that there exists a positive number a such that

$$\lim_{t \to +\infty} |\varphi(t, x, g)| = 0$$

takes place uniformly with respect to $|x| \le a$ and $g \in \Omega_f$. Then the trivial solution of (1) is uniformly asymptotically stable.

The positive solution of G. Sell's conjecture was obtained by Artstein [3] and Bondi et al. [4].

Remark 1.1. 1. Bondi et al. [4] proved this conjecture under the additional assumption that the function f is locally Lipschitzian.

2. Artstein [3] proved this statement without Lipschitzian condition. In reality he proved a more general statement. Namely, he supposed that only limiting equations for (1) are regular, but the function f is not obligatory regular.
3. It is known (see, for example, [5]) that for a wide class of ordinary differential equations (ODEs) the notions of uniform asymptotic stability and the notion of stability in the sense of Duboshin (total stability or stability under the perturbation) are equivalent. There is a series of works (see, for example, [4,6–12] and the references therein), where the authors study the analog of G. Sell's problem for total stability.

In this paper, we will formulate G. Sell's conjecture for the abstract NDS. We will give a positive answer to this conjecture and we will apply this result to different classes of evolution equations: infinite-dimensional differential equations, functional-differential equations and semi-linear parabolic equations.

The paper is organized as follows.

In Section 2, we collect some notions (global attractor, stability, asymptotic stability, uniform asymptotic stability, minimal set, point/compact dissipativity, recurrence, shift dynamical systems, etc.) and facts from the theory of dynamical systems which will be necessary in this paper.

Section 3 is devoted to the analysis of G. Sell's conjecture. In this section, we formulate an analog of G. Sell's conjecture for cocycles and general NDS.

In Section 4, we establish the relation between different types of stability of NDS. We prove that from uniform attractiveness uniform asymptotic stability follows (Theorem 4.1). It is proved that for asymptotically compact dynamical system, asymptotic stability and uniform asymptotic stability are equivalent (Theorem 4.3). The main results of this section are Theorem 4.7 and Corollary 4.9 which contain s positive answer to G. Sell's conjecture for general NDS.

Finally, Section 5 contains some applications of our general results from Sections 2–4 for ODEs (Theorem 5.1), Functional-Differential Equations (Theorem 5.5) and Semi-Linear Parabolic Equations (Theorem 5.7).