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Singular limit and exact decay rate of a nonlinear elliptic equation

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ABSTRACT

For any $n \geq 3$, $0 < m \leq (n-2)/n$, and constants $\eta > 0$, $\beta > 0$, $\alpha \leq \beta(n-2)/m$, we prove the existence of radially symmetric solution of $\frac{n-1}{m}\Delta v^m + \alpha v + \beta x \cdot \nabla v = 0$, $v > 0$, in $\mathbb{R}^n$, $v(0) = \eta$, without using the phase plane method. When $0 < m < (n-2)/n$, $n \geq 3$, we prove that v satisfies $\lim_{|x|\to\infty} \frac{|x|^2 v(x)^{1-m}}{\log |x|} = \frac{2(n-1)(n-2-nm)}{\beta(1-m)}$ if $\alpha = 2\beta/(1-m) > 0$ and $\lim_{|x|\to\infty} |x|^{\alpha/\beta} v(x) = A$ for some constant $A > 0$ if $2\beta/(1-m) > \max(\alpha, 0)$. For $\beta > 0$ or $\alpha = 0$, we prove that the radially symmetric solution $v^{(m)}$ of the above elliptic equation converges uniformly on every compact subset of $\mathbb{R}^n$ to the solution of an elliptic equation as $m \to 0$.

1. Introduction

Recently there is a lot of interest in the following singular diffusion equation [1–3],

$$u_t = \frac{n-1}{m}\Delta u^m \quad \text{in } \mathbb{R}^n \times (0, T) \tag{1.1}$$

which arises in the study of many physical models. When $0 < m < 1$, (1.1) is called the fast diffusion equation. When $m \to 0$, after a rescaling (1.1) becomes

$$u_t = \Delta \log u \quad \text{in } \mathbb{R}^n \times (0, T). \tag{1.2}$$

For

$$0 < m < \frac{n-2}{n}, \quad n \geq 3, \tag{1.3}$$

asymptotic behaviour of the solution of (1.1) near finite extinction time is studied by Galaktionov and Peletier [4], and Daskalopoulos and Sesum [5]. Extinction profile for solutions of (1.1) for the case $m = (n-2)/n$ is studied in [6] by del Pino and Sáez. Behaviour of solutions near extinction time for the solution of (1.2) is studied by Daskalopoulos and del Pino [7]. Interested reader can read the book [2] by Daskalopoulos and Kenig and the book [8] by Vazquez for the most recent results on (1.1).

For any $n \in \mathbb{Z}^+$, $n \geq 3$, $0 < m < 1$, $\eta > 0$, suppose v is the solution of

$$\begin{cases} \dfrac{n-1}{m}\Delta v^m + \alpha v + \beta x \cdot \nabla v = 0, \quad v > 0, \text{ in } \mathbb{R}^n \\ v(0) = \eta. \end{cases} \tag{1.4}$$

Then as observed by Gilding and Peletier [9] and others [8,10,11], the function

$$u_1(x, t) = t^{-\alpha} v(xt^{-\beta})$$

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is a solution of (1.1) in $\mathbb{R}^n \times (0, \infty)$ if

$$\alpha = \frac{2\beta - 1}{1 - m} \tag{1.5}$$

and for any $T > 0$ the function

$$u_2(x, t) = (T - t)^{\alpha} v(x(T - t)^{\beta})$$

is a solution of (1.1) in $\mathbb{R}^n \times (0, T)$ if

$$\alpha = \frac{2\beta + 1}{1 - m} > 0 \tag{1.6}$$

and the function

$$u_3(x, t) = e^{-\alpha t} v(x e^{-\beta t})$$

is an eternal solution of (1.1) in $\mathbb{R}^n \times (-\infty, \infty)$ if

$$\alpha = \frac{2\beta}{1 - m}. \tag{1.7}$$

On the other hand Daskalopoulos and Sesum [10] proved that a locally conformally flat gradient Yamabe soliton with positive sectional curvature must be radially symmetric and the metric $g_{ij} = v^{\frac{4}{n+2}} dx^2$ satisfies (1.4) or

$$\frac{n-1}{m}\left((v^m)'' + \frac{n-1}{r}(v^m)'\right) + \alpha v + \beta r v' = 0, \quad v > 0, \tag{1.8}$$

in $(0, \infty)$ and

$$\begin{cases} v(0) = \eta \\ v'(0) = 0 \end{cases} \tag{1.9}$$

for some constant $\eta > 0$ where dx^2 is the standard metric on $\mathbb{R}^n$ with $m = (n-2)/(n+2)$, $n \ge 3$, and

$$\alpha = \frac{2\beta + \rho_1}{1 - m} \tag{1.10}$$

for some constants $\beta > 0$, α, and ρ_1 where $\rho_1 = 0$ if g_{ij} is a Yamabe steady soliton, $\rho_1 < 0$ if g_{ij} is a Yamabe expander soliton, and $\rho_1 > 0$ if g_{ij} is a Yamabe shrinker soliton.

Since the asymptotic behaviour of the solutions of (1.1) is usually similar to either the functions u_1, u_2 or u_3, it is important to study the solutions of (1.4) in order to understand the behaviour of solutions of (1.1) and the locally conformally flat gradient Yamabe solitons. The existence and uniqueness of radially symmetric solution of (1.4) for α, β, satisfying (1.3) and (1.6) is proved by Peletier, Zhang [12] and King [13] using the phase plane method (cf. Proposition 7.4 of [8]). The existence of radially symmetric solution of (1.4) for $\alpha, \beta > 0$, satisfying (1.3) and (1.5) is proved in p. 22 of [10]. A sketch of the proof of the existence of the radially symmetric solution of (1.4) for $m = (n-2)/(n+2)$, $n \ge 3$, and $\alpha, \beta > 0$, satisfying (1.7) is given in pp. 22–23 of [10]. This existence result is also noted without proof in [14].

In [10] Daskalopoulos and Sesum also proved that if $m = (n-2)/(n+2)$, $n \ge 6$ and $\alpha, \beta > 0$, satisfy (1.7), then the radially symmetric solution of (1.4) satisfies

$$C_1 \frac{\log|x|}{|x|^2} \le v(x)^{1-m} \le C_2 \frac{\log|x|}{|x|^2} \qquad \text{as } |x| \to \infty \tag{1.11}$$

for some constants $C_2 > C_1 > 0$.

In this paper, we will extend the result of [10] and prove the following theorem.

Theorem 1.1. *Let $\eta > 0$ and let $\alpha, \beta \in \mathbb{R}$, n, m, satisfy*

$$0 < m \le \frac{n-2}{n}, \quad n \ge 3, \tag{1.12}$$

and

$$\alpha \le \frac{\beta(n-2)}{m} \quad \text{and} \quad \beta > 0. \tag{1.13}$$

Then there exists a unique solution v of (1.8) and (1.9), in $(0, \infty)$. Moreover for $\alpha \ne 0$, $k = \beta/\alpha$, the function

$$w_1(r) = r^2 v(r)^{2k} \tag{1.14}$$

satisfies $w_1'(r) > 0$ for all $r > 0$.