Nonlinear Analysis 75 (2012) 7–21

Contents lists available at SciVerse ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Asymptotic bifurcation points, and global bifurcation of nonlinear operators and its applications☆

Xiaoying Liu, Jingxian Sun*

Department of Mathematics, Xuzhou Normal University, Xuzhou, 221116, Jiangsu, People's Republic of China

ARTICLE INFO

Article history:
Received 16 January 2010
Accepted 1 March 2011
Communicated by Ravi Agarwal

Keywords:
Asymptotic bifurcation point
Global bifurcation
Topological degree
Superlinear Sturm–Liouville problems

ABSTRACT

Using the cone theory and lattice structure, we discuss the existence of asymptotic bifurcation points and the global bifurcation of nonlinear operators which are not assumed to be cone mappings and may not be Frechet differentiable at points at infinity. As an application, the structure of the set of solutions of the superlinear Sturm–Liouville problems is investigated.

1. Introduction

Let E be a Banach space with norm $\|x\|$ and $A : E \to E$ a completely continuous operator. Obviously, $\mathbb{R} \times E$ becomes a Banach space with norm $\|(\lambda, x)\| = (\lambda^2 + \|x\|^2)^{\frac{1}{2}}$. The closure of the set of nonzero solutions of the equation

$$x = \lambda Ax \tag{1.1}$$

will be denoted by L, i.e.,

$$L = \{(\lambda, x) \in \mathbb{R} \times E | x = \lambda Ax,\ x \neq \theta\}.$$

In this paper, we shall study the existence of asymptotic bifurcation points and the global bifurcation of nonlinear operators. The difference from [1] is that the operator A is not assumed to be a cone mapping and may not be Frechet differentiable at points at infinity. Our results generalize and complement the corresponding results in [2–4].

The degree theory is an effective tool for investigating a lot of nonlinear problems; one could see [5–8] and the references therein. On the other hand, the Nielsen fixed point theory has been applied to some nonlinear equations; see [9,10]. We discuss the global bifurcation by using the topological degree method for the lattice structure. As an application, we investigate the structure of the set of solutions of the superlinear Sturm–Liouville problems.

We use the following lemma.

Lemma 1.1. *Suppose that $A : E \to E$ is a completely continuous operator which is Frechet differentiable at θ and $A\theta = \theta$. Suppose that $\lambda \in \mathbb{R}$, U is a bounded open set in $[\lambda, +\infty) \times E$ (or $(-\infty, \lambda] \times E$), and*

$$\partial U \cap L = \emptyset,$$

where ∂U is the boundary of U relative to $[\lambda, +\infty) \times E$ (correspondingly, $(-\infty, \lambda] \times E$). Then we have:

(i) *if $(\lambda, \theta) \in \underline{U}$, then* $\deg(I - \lambda A, U(\lambda), \theta) \equiv 1 \pmod 2$, *where $U(\lambda) = U \cap (\{\lambda\} \times E)$;*
(ii) *if $(\lambda, \theta) \notin \overline{U}$, then* $\deg(I - \lambda A, U(\lambda), \theta) \equiv 0 \pmod 2$.

☆ The project supported by the National Natural Science Foundation of PR China (10971179, 11171286).
* Corresponding author.
E-mail addresses: liuxy@xznu.edu.cn, xzliuxy2007@yahoo.com.cn (X. Liu), jxsun7083@sohu.com (J. Sun).

0362-546X/$ – see front matter © 2011 Elsevier Ltd. All rights reserved.
doi:10.1016/j.na.2011.03.012

Remark 1.1. The proof of Lemma 1.1 can be found in [11,12], or see [13]. The proof idea came from [4]. We give the proof in Appendix A.

In order to prove the main conclusion of this paper, we need some results from point set topology.

Lemma 1.2 ([14]). *Let M be a compact metric space and A and B be disjoint, closed subsets of M. Then either there exists a closed, connected set C such that $C \cap A$ and $C \cap B$ are nonempty or there exist disjoint, closed subsets M_A and M_B of M such that $A \subset M_A$, $B \subset M_B$, and $M = M_A \cup M_B$.*

Let M be a compact metric space and $\{C_n\}$ a sequence of subsets of M. Define

$$\limsup_{n\to\infty} C_n = \{x \in M \mid \text{ there exist a subsequence } \{n_k\} \text{ of } \{n\} \text{ and } x_{n_k} \in C_{n_k} \text{ such that } \lim_{k\to\infty} x_{n_k} = x\}.$$

Clearly, $\limsup_{n\to\infty} C_n$ is a closed set of M.

In papers [15,16], the author, Sun, proved the following lemma.

Lemma 1.3. *Suppose that M is a compact metric space and $(a, b) \subset (-\infty, +\infty)$ (where a and b may be $-\infty$ and $+\infty$ respectively), and*

$$a < \cdots < \alpha_n < \cdots < \alpha_2 < \alpha_1 < \beta_1 < \beta_2 < \cdots < \beta_n < \cdots < b,$$
$$\lim_{n\to\infty} \alpha_n = a, \qquad \lim_{n\to\infty} \beta_n = b.$$

Suppose that $\Sigma = \{C_n \mid n = 1, 2, \cdots\}$ is a family of connected subsets of $\mathbb{R} \times M$, which satisfies the following conditions:

(1) *for $n = 1, 2, \ldots,$*

$$C_n \cap (\{\alpha_n\} \times M) \neq \emptyset,$$
$$C_n \cap (\{\beta_n\} \times M) \neq \emptyset;$$

(2) *for $a < \alpha < \beta < b$, $\left(\bigcup_{n=1}^{\infty} C_n\right) \bigcap([\alpha, \beta] \times M)$ is a relatively compact set of $\mathbb{R} \times M$.*

Then there exists a connected component C^* of $\limsup_{n\to\infty} C_n$ such that

$$C^* \cap (\{\lambda\} \times M) \neq \emptyset, \quad \forall \lambda \in (a, b).$$

We need the following lemma which is a version of the lemma above; the proof is similar to that in [15,16]—see Appendix B.

Lemma 1.4. *Let E be a Banach space, and $\{C_n\}$ a sequence of connected subsets of $\mathbb{R} \times E$ satisfying:*

(1) *there exists $\bar{\lambda} \in \mathbb{R}$ such that $\left(\bigcup_{n=1}^{\infty} C_n\right) \cap (\{\bar{\lambda}\} \times E)$ is a bounded set, and for any $R > 0$, $\left(\bigcup_{n=1}^{\infty} C_n\right) \cap S_R$ is a relatively compact set of $\mathbb{R} \times E$, where $S_R = \{(\lambda, x) \in \mathbb{R} \times E \mid \|(\lambda, x)\| < R\}$;*

(2) *there exists $\{R_n\} \subset \mathbb{R}$ with*

$$R_1 < R_2 < \cdots < R_n < \cdots, \quad R_n \to +\infty (n \to \infty),$$

and $R_1 > \sup\{\|(\lambda, x)\| \mid (\lambda, x) \in \left(\bigcup_{n=1}^{\infty} C_n\right) \cap (\{\bar{\lambda}\} \times E)\}$ such that

$$C_n \cap (\partial S_{R_n} \cap ((-\infty, \bar{\lambda}) \times E)) \neq \emptyset, \quad n = 1, 2, \ldots,$$
$$C_n \cap (\partial S_{R_n} \cap ((\bar{\lambda}, +\infty) \times E)) \neq \emptyset, \quad n = 1, 2, \ldots,$$

where ∂S_{R_n} is the boundary of S_{R_n} in $\mathbb{R} \times E$.

Then there exists a connected component C of $\limsup_{n\to\infty} C_n$ such that for $R > R_1$,

$$C \cap (\partial S_R \cap ((-\infty, \bar{\lambda}) \times E)) \neq \emptyset; \qquad C \cap (\partial S_R \cap ((\bar{\lambda}, +\infty) \times E)) \neq \emptyset. \tag{1.2}$$

2. General results

Theorem 2.1. *Suppose that E is a Banach space, $A : E \to E$ is a completely continuous operator which is Frechet differentiable at θ and $A\theta = \theta$. Suppose that there exists $\bar{\lambda} > 0$ such that*

$$\text{ind}(I - \bar{\lambda} A, \infty) \equiv 0 (\text{mod } 2), \tag{2.1}$$

i.e., for sufficiently large $R > 0$, the topological degree $\deg(I - \bar{\lambda} A, B_R, \theta) \equiv 0 \ (\text{mod } 2)$, where $B_R = \{x \in E \mid \|x\| < R\}$. Then L possesses an unbounded connected component $C \subset (0, +\infty) \times E$, which satisfies:

(i) *there exists an asymptotic bifurcation point λ^* of A with $\lambda^* \in [0, \bar{\lambda}]$ such that C passes through (λ^*, ∞), i.e., for any $\delta > 0$, $M > 0$, there exists $(\lambda, x) \in C$ such that $|\lambda - \lambda^*| < \delta$, $\|x\| > M$;*

(ii) *$C \cap ((0, \bar{\lambda}) \times E)$ is unbounded;*

(iii) *$C \cap ((\bar{\lambda}, +\infty) \times E)$ is unbounded.*