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The Denjoy–Wolff theorem in $\mathbb{C}^n$

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This paper is dedicated to Professor Kazimierz Goebel on his 70th birthday

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ABSTRACT

If D is a bounded and strictly convex domain in $\mathbb{C}^n$ and $f : D \to D$ is holomorphic and fixed-point-free, then there exists $\tilde{\xi} \in \partial D$ such that the sequence $\{f^n\}$ of the iterates of f converges in the compact–open topology to the constant mapping taking the value $\tilde{\xi}$.

1. Introduction

Let D be a bounded and strictly convex domain in $\mathbb{C}^n$ and let $f : D \to D$ be a holomorphic and fixed-point-free self-mapping. In view of the Denjoy–Wolff theorem [1–4] it is natural to ask whether the sequence of iterates $\{f^n(x)\}$, where $x \in D$, converges. There are many papers on the problem of iterating holomorphic and fixed-point-free mappings in $\mathbb{C}^n$; see, *e.g.*, [5–26] for interesting surveys and references. One of the most well known results in $\mathbb{C}^n$ is due to Abate [5].

Theorem 1.1. *If $D \subset \mathbb{C}^n$ is a bounded and strongly convex domain with a C^2 boundary and $f : D \to D$ is holomorphic and fixed-point-free, then there exists $\xi \in \partial D$ such that the sequence $\{f^n\}$ of iterates of f converges in the compact–open topology to the constant map taking the value ξ.*

For the definition of strong convexity see, for example, [7].

In this paper we extend the Abate result to strictly convex domains in $\mathbb{C}^n$.

Note that in the infinite dimensional case it is known [27] that the Denjoy–Wolff theorem fails even for biholomorhic self-maps of the unit ball. But if f is either a condensing holomorphic self-map of a strictly convex unit ball, a holomorphic automorphism of a special type, a firmly k_B-nonexpansive mapping, or an averaged mapping of the first or second kind, then positive results were established in [28–40,26,41] (see also [42,43] for compact holomorphic self-mappings of the open unit balls of J^*-algebras, and [44,45] for analytic functions of operators).

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2. Preliminaries

If D is a bounded and convex domain in $\mathbb{C}^n$, then k_D always denotes its Kobayashi distance [46–52]. We recall the definition of the Kobayashi distance. Let Δ be the open unit disc in the complex plane $\mathbb{C}$. The Poincaré distance on Δ is given by

$$k_\Delta(z, w) = \rho_\Delta(z, w) = \arg\tanh \left| \frac{z - w}{1 - z\overline{w}} \right| = \arg\tanh(1 - \sigma(z, w))^{\frac{1}{2}},$$

where

$$\sigma(z, w) = \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - z\overline{w}|^2}, \quad z, w \in \Delta.$$

Now, let D be a bounded convex domain in $\mathbb{C}^n$. We will use the definition of the Kobayashi distance (see [48]) which is due to Lempert (see [51]):

$$k_D(x, y) = \delta_D(x, y) = \inf\{k_\Delta(0, \gamma) : \text{there exists } f \in H(\Delta, D) \text{ such that } f(0) = x \text{ and } f(\gamma) = y\}.$$

The function δ is also called the Lempert function.

One can check that the Kobayashi distance k_D is locally equivalent to the norm $\| \cdot \|_2$ in $\mathbb{C}^n$ (see [30]).

The following concept of a complex geodesic will play a key role in our considerations.

Definition 2.1 (*Compare [52]*)**.** Let D be a bounded and convex domain in $\mathbb{C}^n$. A holomorphic mapping $\phi : \Delta \to D$ is a complex geodesic (with respect to k_D) if there exist points $z \neq w$ in Δ such that

$$k_\Delta(w, z) = k_D(\phi(w), \phi(z)).$$

Then we say that $\phi(w)$ and $\phi(z)$ are joined by a complex geodesic. If, moreover, $w = 0$ and $0 < z \in \mathbb{R}$, we call ϕ a normalized complex geodesic joining $\phi(w)$ with $\phi(z)$.

Theorem 2.1 (*Compare [53]*)**.** *Let D be a bounded and convex domain in $\mathbb{C}^n$. Then every pair of distinct points in D can be joined by a complex k_D-geodesic.*

We have the following properties of the Kobayashi distance.

Lemma 2.2 (*Compare [33,54,36,55]*)**.** *Let D be a bounded and convex domain in $\mathbb{C}^n$.*

(a) *If $x, y, w, z \in D$ and $s \in [0, 1]$, then*

$$k_D(sx + (1 - s)y, sw + (1 - s)z) \leq \max[k_D(x, w), k_D(y, z)].$$

(b) *If $x, y \in D$ and $s, t \in [0, 1]$, then*

$$k_D(sx + (1 - s)y, tx + (1 - t)y) \leq k_D(x, y).$$

A subset C of a bounded domain D is said to lie strictly inside D if $dist_{\|\cdot\|}(C, \partial D) = \inf\{\|x - y\| : x \in C,\ y \in \partial D\} > 0$. A mapping $f : D \to D$ is said to map D strictly inside D if $f(D)$ lies strictly inside D.

Theorem 2.3 (*Compare [56]*)**.** *Let D be a bounded and convex domain in $\mathbb{C}^n$. A subset C of D is k_D-bounded if and only if C is strictly inside D.*

Now we need the following definition.

Definition 2.2 (*Compare [57]*)**.** We say that a bounded and convex domain D in $\mathbb{C}^n$ is strictly convex if for every $x, y \in \overline{D}^{\|\cdot\|}$ the open segment

$$(x, y) = \{z \in X : z = sx + (1 - s)y \text{ for some } 0 < s < 1\}$$

lies in D.

Using strict convexity one can prove the following very useful theorem and lemma.

Theorem 2.4 (*Compare [53]*)**.** *Let D be a bounded and strictly convex domain in $\mathbb{C}^n$. Then any pair of distinct points in D can be joined by a unique normalized k_D-geodesic.*

Lemma 2.5 (*Compare [33]*)**.** *Let D be a bounded and strictly convex domain in $\mathbb{C}^n$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in D which converge to $\xi \in \partial D$ and to $\eta \in \overline{D}$, respectively. If*

$$\sup\{k_D(x_n, y_n) : n = 1, 2, \ldots\} = c < \infty,$$

then $\xi = \eta$.

The next theorem is called the Earle–Hamilton theorem [58] (see also [53,59,30,25]).