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Lie point symmetries, partial Noether operators and first integrals of the Painlevé–Gambier equations

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ABSTRACT

We utilize the Lie–Tressé linearization method to obtain linearizing point transformations of certain autonomous nonlinear second-order ordinary differential equations contained in the Painlevé–Gambier classification. These point transformations are constructed using the Lie point symmetry generators admitted by the underlying Painlevé–Gambier equations. It is also shown that those Painlevé–Gambier equations which have a few Lie point symmetries and hence are not linearizable by this method can be integrated by a quadrature. Moreover, by making use of the partial Lagrangian approach we obtain time dependent and time independent first integrals for these Painlevé–Gambier equations which have not been reported in the earlier literature. A comparison of the results obtained in this paper is made with the ones obtained using the generalized Sundman linearization method.

1. Introduction

The nonlinear second-order ordinary differential equation (ODE),

$$\ddot{x}+\frac{1}{2}\phi_x\dot{x}^2+\phi_t\dot{x}+B(t,x)=0, \tag{1.1}$$

where $x=x(t)$, $\phi=\phi(t,x)$, $\dot{x}=\frac{dx}{dt}$ and so on, is known as the Jacobi equation and was considered in the classical work of Jacobi [1] and recently in the works of [2,3].

In [2], Lagrangians of Eq. (1.1) were derived for this class of equations by using the method of Jacobi [1]. In [3], Guha et al. investigated generalized Sundman transformations and Sundman symmetries of this class of equations which was also contained in the Painlevé–Gambier classification [4–6]. They obtained first integrals for some equations in this class.

In this paper, we revisit Eq. (1.1) from the view point of Lie point symmetries and partial Lagrangians. In particular, we compare the Lie linearization of the class of Eq. (1.1) with the Sundman linearization and show that for $\phi_t=0$ and $B=0$, the Lie and Sundman linearizations are the same. In the case, $\phi_t=0$ and $B_t=0$, we obtain few symmetries from which we obtain the first integrals. We compare our results with the results of Guha et al. [3]. Then we also obtain partial Noether operators and first integrals of the subclasses of the class of Eq. (1.1).

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The outline of the paper is as follows. In Section 2, we present the conditions of Lie–Tressé (see, e.g. Mahomed [7] and the references therein) for the linearization by point transformation of the class of Eq. (1.1). Here we also consider as special cases of the class of Eq. (1.1), particular equations contained in the Painlevé–Gambier classification for the Lie linearization. The equations in the subclass of (1.1) which are not linearizable and have a few Lie point symmetry generators are shown to be integrable by a quadrature. In Section 3, by using the partial Lagrangian method we obtain new first integrals for certain linearizable and nonlinearizable equations which are special cases of the class of Eq. (1.1) that are considered in Section 2. Finally, in Section 4, we compare our results with those obtained in [3] via generalized Sundman transformations.

2. Lie linearization

We consider in this section, the linearizability of Eq. (1.1) by point transformation. We use the invariant criteria of Lie–Tressé [7]. By using the linearizing condition 5 as given in Theorem 8 in [7] to the class of Eq. (1.1), we find that the first of the conditions is satisfied, and the second one provides the following relationship between ϕ and B given by

$$\phi_{ttx} + \phi_t\phi_{tx} - B\phi_{xx} - B_x\phi_x - 2B_{xx} = 0. \tag{2.1}$$

We now look at the two cases, viz., $\phi_t = 0,\ B = 0$ and $\phi_t = 0,\ B_t = 0$. These cases were previously considered in Guha et al. [3]. However, they used Sundman transformations.

Case 1. $\phi_t = 0 = B$.

In this case, it is obvious that (2.1) is satisfied. Hence the corresponding equation arising from (1.1) is linearizable by a point transformation. We now present some specific examples for the linearization of this class of equations which were contained in the Painlevé–Gambier classification and also considered in [3].

(a) Painlevé–Gambier equation XI.

We first consider the Painlevé–Gambier equation XI

$$\ddot{x} - \frac{\dot{x}^2}{x} = 0. \tag{2.2}$$

Eq. (2.2) has the maximum eight-dimensional Lie algebra. The two non-commuting symmetry generators are

$$X_1 = tx\partial_x, \qquad X_2 = -t\partial_t. \tag{2.3}$$

By condition 9 in Theorem 8 in [7], we find the linearizing transformation $X = X(t,x),\ Y = Y(t,x)$ that will reduce X_1 and X_2 to their canonical form

$$X_1 = \partial_Y, \qquad X_2 = X\partial_X + Y\partial_Y. \tag{2.4}$$

Thus, by solving the system of partial differential equations

$$X_1(X) = 0, \qquad X_2(X) = X, \tag{2.5}$$

$$X_1(Y) = 1, \qquad X_2(Y) = Y, \tag{2.6}$$

we obtain the following linearizing point transformation

$$X(t,x) = \frac{1}{t}, \qquad Y(t,x) = \frac{\ln x}{t}, \tag{2.7}$$

which linearizes Eq. (2.2) to the equation $Y'' = 0$.

Remark 1. The authors in [3], used the generalized Sundman transformation $X = F(t,x) = (\ln x/t)^2,\ G(t,x) = 2\ln x/t^3$ to linearize Eq. (2.2) to $X'' = 0$. The above example clearly illustrates the possibility of the Lie linearization of (2.2) by point transformation as well.

(b) Painlevé–Gambier equation XXXVII.

Next we consider the Painlevé–Gambier equation XXXVII

$$\ddot{x} - \left(\frac{1}{2x} + \frac{1}{x-1}\right)\dot{x}^2 = 0. \tag{2.8}$$

Eq. (2.8) has Lie algebra spanned by eight Lie point symmetry generators. The two non-commuting symmetry generators are

$$X_1 = t\sqrt{x}(x-1)\partial_x, \qquad X_2 = -t\partial_t. \tag{2.9}$$

Now the procedure of finding the linearizing transformation of Eq. (2.8) is the same as in the above example, thus by invoking condition 9 in Theorem 8 in [7], we find the following linearizing transformation

$$X = \frac{1}{t}, \qquad Y = \frac{1}{t}\ln\left(\frac{\sqrt{x}-1}{\sqrt{x}+1}\right) \tag{2.10}$$

of Eq. (2.8) to the equation $Y'' = 0$.