Nonlinear Analysis 75 (2012) 55–67

Contents lists available at SciVerse ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

# Positive solutions of semilinear biharmonic equations with critical Sobolev exponents☆

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ARTICLE INFO

*Article history:*
Received 18 March 2011
Accepted 31 July 2011
Communicated by S. Ahmad

*MSC:*
35J35
35J40
58E30

*Keywords:*
Critical Sobolev exponent
Biharmonic equation
Positive solutions

ABSTRACT

In this paper we study the critical growth biharmonic problem with a parameter $\lambda$ and establish uniform lower bounds for $\Lambda$, which is the supremum of the set of $\lambda$, related to the existence of positive solutions of the biharmonic problem.



## 1. Introduction

For many years the following semilinear second-order elliptic problem:

$$\begin{cases} -\Delta u = f_\lambda(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

has been extensively studied; see e.g. [1–5] and the references therein. Here, $\Omega$ is a bounded smooth domain in $\mathbb{R}^N$, $f_\lambda : \Omega \times \mathbb{R} \to \mathbb{R}$ and $\lambda$ is a real parameter. When $f_\lambda$ is sublinear, for example, $f_\lambda = \lambda u^q,\ 0 < q < 1$, sub-solutions and super-solutions easily provide the existence of a unique positive solution of (1.1) for all $\lambda > 0$ by using a monotone iteration scheme; see [6, Theorem 2.3.1]. And when $f_\lambda$ is superlinear, say $f_\lambda = \lambda u + |u|^{p-1}u,\ 1 < p < \frac{N+2}{N-2}$, variational methods are quite convenient; see [7]. When $p$ equals the critical Sobolev exponent $\frac{N+2}{N-2}$, the problem becomes delicate because of the lack of compactness. Since the functional corresponding to (1.1) does not satisfy the Palais–Smale condition, there are serious difficulties when trying to find critical points by standard variational methods; see [4]. Ambrosetti et al. [8] considered, in contrast with the pure sublinear case and the superlinear case, (1.1) when $f_\lambda$ is, roughly, the sum of a sublinear and a superlinear term. To be precise, they considered the following problem:

$$\begin{cases} -\Delta u = \lambda u^q + u^p & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

☆ This research project was supported by NNSFC (No. 60974034), NSF of Shanxi Province (No. 2009011008), Shanxi Scholarship Council of China (No. 2011-005) and TYAL.

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0362-546X/$ – see front matter © 2011 Elsevier Ltd. All rights reserved.
doi:10.1016/j.na.2011.07.065

with $0 < q < 1 < p$. They proved that (1.2) admits a positive solution if and only if $0 < \lambda \leq \Lambda$ for a suitable positive constant $\Lambda \in \mathbb{R}$. The proof was based on the method of sub-solutions and super-solutions. In [9] Sun and Li considered the following problem:

$$\begin{cases} -\Delta u = u^q + \lambda u^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

with $0 < q < 1 < p = \frac{N+2}{N-2}$. The authors also studied the value of $\Lambda$-the supremum of the set of $\lambda$, related to the existence and multiplicity of positive solutions of (1.3) and established uniform lower bounds for $\Lambda$, that is,

$$\Lambda \geq \frac{1-q}{p-q}\left(\frac{p-1}{p-q}\right)^{(p-1)/(1-q)} S_N^{\frac{N}{N-2}} \left[\inf_{u\in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2}{\left(\int_\Omega |u|^{1+q}\right)^{\frac{2}{1+q}}}\right]^{\frac{(1+q)(p-1)}{2(1-q)}},$$

where $S_N$ is the best constant of the Sobolev imbedding $H_0^1(\Omega) \hookrightarrow L^{2N/(N-2)}(\Omega)$.

Some interesting generalizations of (1.2) have been considered in the framework of second-order quasi-linear operators or biharmonic operators. We refer the reader to [10] for equations associated with the $p$-Laplace operator and to [11] for the case of the biharmonic operator. Recently the fourth-order problems have been studied by many authors; we refer the reader to [11–14] for some references. Following this trend, we consider the following semilinear biharmonic problem:

$$\begin{cases} \Delta^2 u = \lambda|u|^{q-1}u + |u|^{p-1}u & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

with $0 < q < 1 < p = \frac{N+4}{N-4}$, $N > 4$. To the author's knowledge, there seem few results on problem (1.4) for the case $0 < q < 1$. In [11] Bernis et al. proved the following result among other things.

**Theorem A.** *Assume that $0 < q < 1 < p = \frac{N+4}{N-4}$. Then there exists a constant $\Lambda > 0$ such that problem (1.4) admits at least a positive solution if $0 < \lambda < \Lambda$ and no positive solutions if $\lambda > \Lambda$.*

We are interested in finding the values of $\Lambda$ at which the above transition occurs. Actually we obtain the lower bounds for $\Lambda$. Before stating our main result, we give some notation. The norm of $u$ in $L^r(\Omega)$ is denoted by $|u|_r = \left(\int_\Omega |u(x)|^r dx\right)^{\frac{1}{r}}$; $H$ denotes $H_0^1(\Omega)\cap H^2(\Omega)$, endowed with the norm $\|u\| = |\Delta u|_2$; $C_1, C_2, C_3, \ldots$ denote (possibly different) positive constants. By $O(t)$, $o(t)$ we mean $|O(t)| \leq Ct$, $\frac{o(t)}{t} \to 0$, as $t \to 0$ respectively. $o(1)$ will be used to denote quantities that tend to 0.

Our main result is as follows.

**Theorem 1.1.**

$$\Lambda \geq \lambda^* := \frac{p-1}{p-q}\left(\frac{1-q}{p-q}\right)^{\frac{1-q}{p-1}} S^{\frac{(1-q)(p+1)}{2(p-1)}} \left(\inf_{u\neq 0} \frac{\|u\|^2}{|u|_{q+1}^2}\right)^{\frac{q+1}{2}},$$

*where $S$ corresponds to the best constant for the Sobolev embedding $H \hookrightarrow L^{p+1}(\Omega)$.*

For $u \in H$ define

$$J_\lambda(u) = \frac{1}{2}\|u\|^2 - \frac{\lambda}{q+1}|u|_{q+1}^{q+1} - \frac{1}{p+1}|u|_{p+1}^{p+1}.$$

Define

$$M_\lambda = \{u \in H : (J_\lambda'(u), u) = 0\},$$

where $(\cdot,\cdot)$ denotes the usual scalar product in $H$. Using an approach similar to the method used in [15], we split $M_\lambda$ into three parts:

$$M_\lambda^+ = \{u \in M_\lambda : (p-1)\|u\|^2 > \lambda(p-q)|u|_{q+1}^{q+1}\},$$

$$M_\lambda^0 = \{u \in M_\lambda : (p-1)\|u\|^2 = \lambda(p-q)|u|_{q+1}^{q+1}\},$$

$$M_\lambda^- = \{u \in M_\lambda : (p-1)\|u\|^2 < \lambda(p-q)|u|_{q+1}^{q+1}\}.$$

Note that all solutions of (1.4) are clearly in $M_\lambda$. Hence our approach to this problem is to study the structure of the set $M_\lambda$. In fact, a norm gap exists between $M_\lambda^+$ and $M_\lambda^-$ when $\lambda$ belongs to some interval (see Lemma 2.2 in Section 2 for details). This phenomenon has been observed in [9] when the authors considered problem (1.3). This feature is crucial in the proof of Theorem 1.1.

The paper is organized as follows: in the next section, we give some lemmas; in Section 3, we prove Theorem 1.1.

## 2. Preliminary results

In this section, we prove several lemmas.

**Lemma 2.1.** *If $\lambda < \lambda^*$, then $M^\pm \neq \emptyset$ and $M_\lambda^0 = \{0\}$.*