Nonlinear Analysis 75 (2012) 78–90

Contents lists available at SciVerse ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Nonuniform (μ, ν)-dichotomies and local dynamics of difference equations

António J.G. Bento *, César M. Silva

Departamento de Matemática, Universidade da Beira Interior, 6201-001 Covilhã, Portugal

ARTICLE INFO

Article history:
Received 11 May 2011
Accepted 1 August 2011
Communicated by Ravi Agarwal

MSC:
37D10
34D09
37D25

Keywords:
Invariant manifolds
Nonautonomous difference equations
Nonuniform generalized dichotomies

ABSTRACT

We obtain a local stable manifold theorem for perturbations of nonautonomous linear difference equations possessing a very general type of nonuniform dichotomy, possibly with different growth rates in the uniform and nonuniform parts. Note that we consider situations where the classical Lyapunov exponents can be zero. Additionally, we study how the manifolds decay along the orbit of a point as well as the behavior under perturbations and give examples of nonautonomous linear difference equations that admit the dichotomies considered.

1. Introduction

The main purpose of this paper is to discuss, in a Banach space X, the existence of stable manifolds for a general family of perturbations of nonautonomous linear difference equation

$$x_{m+1} = A_m x_m + f_m(x_m), \quad m \in \mathbb{N},$$

assuming that the perturbations $f_m : X \to X$ verify $f_m(0) = 0$,

$$\|f_m(u) - f_m(v)\| \le c\|u - v\|(\|u\| + \|v\|)^q, \quad m \in \mathbb{N},$$

for some constants $c > 0$ and $q > 1$ and for each $u, v \in X$, and that the linear equation

$$x_{m+1} = A_m x_m, \quad m \in \mathbb{N},$$

admits a very general type of nonuniform dichotomy given by arbitrary rates of growth.

The notion of uniform exponential dichotomy was introduced by Perron in [1] and constitutes a very important tool in the theory of difference and differential equations, particularly in the study of invariant manifolds. In spite of being used in a wide range of situations, sometimes this notion is too demanding and it is of interest to consider more general kinds of hyperbolic behavior. A much more general type of dichotomy, allowing the rates of growth to vary along the trajectory of a point, is the notion of nonuniform exponential dichotomy that was introduced by Barreira and Valls in the context of nonautonomous differential equations in [2] and that was inspired both in Perron's classical notion of exponential dichotomy

* Corresponding author. Fax: +351 275 319 732.
E-mail addresses: bento@ubi.pt (A.J.G. Bento), csilva@ubi.pt (C.M. Silva).
URL: http://www.mat.ubi.pt/~csilva (C.M. Silva).

doi:10.1016/j.na.2011.08.008

and in the notion of nonuniformly hyperbolic trajectory introduced by Pesin in [3–5]. In the context of difference equations, a notion of nonuniform exponential dichotomy was also introduced in [6].

The study of stable manifolds in the nonuniform context has a long history, starting with a famous theorem on the existence of stable manifolds for nonuniformly hyperbolic trajectories, in the finite dimensional setting, proved by Pesin [3]. In [7], Ruelle gave an alternative proof of this theorem based on the study of perturbations of products of matrices occurring in Oseledets' multiplicative ergodic theorem [8] and, inspired by the classical work of Hadamard, in [9] Pugh and Shub proved the same result using graph transform techniques. In Hilbert spaces and under some compactness assumptions, Ruelle [10] obtained a version of the stable manifold theorem, following his approach in [7]. Versions of this theorem for transformations in Banach spaces, were first established by Mañé [11] under some compactness and invertibility assumptions and then by Thieullen [12] under a weaker hypothesis.

Stable manifolds were also obtained for perturbations of nonautonomous linear differential equations and for perturbations of nonautonomous linear difference equations, assuming respectively that the linear differential equation and linear difference equation admit a nonuniform exponential dichotomy. We refer the reader to the book [13], where the stable manifolds for perturbations of linear differential equations admitting the existence of nonuniform exponential dichotomies are obtained, and also to [6,14,15] for a related discussion in the context of difference equations.

Recently, invariant stable manifolds were obtained for perturbations of nonautonomous linear difference and differential equations, assuming the existence of nonuniform dichotomies that are not exponential. In particular, in the discrete time setting, assuming the existence of a type of polynomial dichotomy for a nonautonomous linear difference equation, the existence of local stable manifolds for a certain class of perturbations was established in [16] and, for a more restricted class, global stable manifolds were also obtained.

Our result can be seen as a discrete counterpart of the results obtained in [17] for nonautonomous differential equations and we emphasize that the stable manifold theorem for perturbations of linear difference equations with nonuniform exponential dichotomies in [6] is included in our theorem as a very particular case and our result also includes as particular cases the stable manifold theorems for polynomial dichotomies, as well as many other situations where the classical Lyapunov exponent is zero. We stress that, to the best of our knowledge, in the context of perturbations of nonautonomous linear difference equations that admit a non-exponential nonuniform dichotomy, our result is the first one addressing the existence of local stable manifolds for the general class of perturbations above. In particular, it is new even for nonuniform polynomial dichotomies (in [16] the polynomial case was already considered, but the type of nonuniform dichotomies considered were different from the ones considered here). In the context of differential equations and under the existence of nonuniform polynomial dichotomies, local and global stable manifolds were also obtained in [18].

As mentioned above, the type of dichotomies considered in this paper are very general, allowing different rates of growth for the uniform and the nonuniform parts and thus, to establish the existence of stable manifolds, we must assume conditions relating the rate of decay of some balls in the stable spaces and the growth rates.

To highlight the generality of this concept of dichotomy, we discuss some families of new examples that verify the hypothesis in our main result. Additionally, we obtain an upper bound for the decay of solutions along the stable manifolds and we study how the stable manifolds vary with the perturbations by giving bounds, in some suitable metric, on the distances between the functions whose graphs are the stable manifolds.

The content of the paper is as follows. In Section 2, we introduce some notation, the main definitions and state the main theorem. In Section 3, we present some examples. In Section 4, we prove the main theorem and finally, in Section 5, we study how the manifolds obtained vary with the perturbations considered.

2. Main result

We say that an increasing sequence $\mu = (\mu_n)_{n\in\mathbb{N}_0}$ is a *growth rate* if $\mu_0 \geq 1$ and $\lim_{n\to+\infty} \mu_n = +\infty$.

Let $\mu = (\mu_n)_{n\in\mathbb{N}_0}$ and $\nu = (\nu_n)_{n\in\mathbb{N}_0}$ be growth rates and let $B(X)$ be the space of bounded linear operators in a Banach space X. Given a sequence $(A_n)_{n\in\mathbb{N}}$ of invertible operators of $B(X)$ and putting

$$\mathcal{A}_{m,n} = \begin{cases} A_{m-1}\cdots A_n & \text{if } m > n, \\ \mathrm{Id} & \text{if } m = n, \end{cases}$$

we say that the linear difference equation

$$x_{m+1} = A_m x_m, \quad m \in \mathbb{N} \tag{1}$$

admits a *nonuniform* (μ, ν)*-dichotomy* if there exist projections P_m, $m \in \mathbb{N}$, such that

$$P_m \mathcal{A}_{m,n} = \mathcal{A}_{m,n} P_n, \quad m, n \in \mathbb{N},$$

and constants $a < 0 \leq b$, $\varepsilon \geq 0$ and $D \geq 1$ such that for every $n \in \mathbb{N}$ and every $m \geq n$,

$$\|\mathcal{A}_{m,n} P_n\| \leq D \left(\frac{\mu_m}{\mu_{n-1}}\right)^a \nu_{n-1}^{\varepsilon}, \tag{2}$$

$$\|\mathcal{A}_{m,n}^{-1} Q_m\| \leq D \left(\frac{\mu_{m-1}}{\mu_n}\right)^{-b} \nu_{m-1}^{\varepsilon}, \tag{3}$$