Nonlinear Analysis 75 (2012) 103–116

Contents lists available at SciVerse ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Radial solutions for a prescribed mean curvature equation with exponential nonlinearity

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ARTICLE INFO

Article history:
Received 8 April 2011
Accepted 4 August 2011
Communicated by Enzo Mitidieri

MSC:
35J93
34C23
47H11

Keywords:
Gelfand problem
Prescribed mean curvature equation
Time map
Topological degree
Radial solution
Exponential nonlinearity

ABSTRACT

We establish an existence result for radial solutions for a prescribed mean curvature equation with exponential nonlinearity. Our methods are based on degree theory combined with a time map analysis. We also obtain two nonexistence results for positive solutions for more general f; one of them is not limited to radial solutions.

1. Introduction

Consider the following problem:

$$\begin{cases} -\mathrm{div}\left(\dfrac{Du}{\sqrt{1+|Du|^2}}\right) = \lambda f(x,u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\lambda > 0$ and Ω is a bounded domain in $\mathbb{R}^N$. When $f(x,u) = \mathrm{e}^u$, (1.1) can be viewed as a variant of the Liouville–Bratu–Gelfand problem.

The classical Liouville–Bratu–Gelfand problem is concerned with positive solutions of the problem

$$\begin{cases} -\Delta u = \lambda \mathrm{e}^u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.2}$$

where Ω is a bounded domain in $\mathbb{R}^N$. The equation arises from a model of combustible gas dynamics. Solutions of (1.2) are steady states for the thermal reaction process. Here $\lambda > 0$ is known as the Frank-Kamenetskii parameter. The existence of

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0362-546X/$ – see front matter
doi:10.1016/j.na.2011.08.010

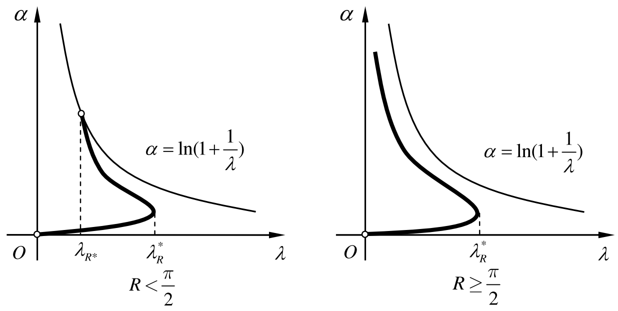

Fig. 1. Bifurcation diagrams for $f(u) = e^u$ in one dimension.

solutions depending on λ and N has been well studied: for the case $N = 1$ by Liouville [1], for $N = 2$ by Bratu [2], for $N = 3$ by Frank-Kamenetskii [3] and Gelfand [4], for all N by Joseph and Lundgren [5]. Radial solutions of Liouville–Bratu–Gelfand type problems with the p-Laplace operator and the k-Hessian operator have been well investigated by Jacobsen and Schmitt [6] (also see the survey [7]).

Nonparametric prescribed mean curvature type equations like (1.1) have been widely investigated by a number of authors [8–23] in recent years. Most of them focus on the case in which the nonlinearity $f(u)$ is chosen to be various power growth functions, while the Liouville–Bratu–Gelfand problem for the mean curvature operator has been neglected for a long time. Recently, for the one-dimensional case, Pan [20] obtained the exact multiplicity of solutions. Moreover, he found an interesting phenomenon: that bifurcation patterns may depend, in some circumstances, on the length of the interval (see Fig. 1 or Lemma 3.2, where $\alpha = \max u = u(0)$).

However, for the higher-dimensional case, the problem is more difficult and still open. At first, the problem is supercritical and standard methods are invalid. Next, the mean curvature operator lacks good structure, unlike the p-Laplace operator or the k-Hessian operator, which have the homogeneity property. Another point is—like what the results for the one-dimensional case imply—that the size of the domain may play an important role in the existence of solutions (also see [20, 21,9]).

In this paper, we consider radial solutions on a ball $\mathcal{B}_R(0)$ in $\mathbb{R}^N$ $(N \geqslant 2)$. The problem (1.1) with $f(x, u) = e^u$ in the radial case takes the form

$$\begin{cases} -\left(\dfrac{r^{N-1}u'(r)}{\sqrt{1+u'(r)^2}}\right)' = r^{N-1}\lambda e^{u(r)}, \quad r \in (0, R), \\ u'(0) = 0 = u(R). \end{cases} \tag{1.3}$$

We obtain an existence result for radial solutions for (1.3). Our methods are based on degree theory combined with a time map analysis. We also obtain two nonexistence results for positive solutions for more general f; one of them is not limited to radial solutions. However, the Liouville–Bratu–Gelfand problem for the mean curvature operator is still far from being completely solved.

The main results of this paper are the following theorems.

Denote by λ_R^* the maximum value of the bifurcation parameter λ of the corresponding one-dimensional problem (see Fig. 1 or Lemma 3.2).

Theorem 1.1. *If $0 < \lambda < \lambda_R^*$, then the problem (1.3) has at least one positive solution in $C^2[0, R]$.*

The result is motivated by Capietto et al. [10], where the nonlinearity f is sublinear and $f(|x|, 0) = 0$. The proof of Theorem 1.1 is based on topological degree methods used in [10] and a detailed time map analysis shown in [20].

Remark 1. We note that the barrier method of Noussair et al. [17] is suitable for application to the problem (1.1), but the result obtained by [17, Thm 2.1] is different from Theorem 1.1 (see Section 5 for a detailed comparison).

Remark 2. Since the problem (1.1) is quasilinear and non-uniformly elliptic, many successful approaches to the semilinear case cannot applied directly to (1.1) due to the lack of gradient estimates. For instance, the barrier method in [17] has to have an additional Assumption (B) to obtain the iterative sequence (see Section 5). For the results of the semilinear case on general bounded domains, see [24, Thm. 2.1] or [25].

The following is a nonexistence result for radial positive solutions to the problem

$$\begin{cases} -\left(\dfrac{r^{N-1}u'(r)}{\sqrt{1+u'(r)^2}}\right)' = r^{N-1}\lambda f(r, u(r)), \quad r \in (0, R), \\ u'(0) = 0 = u(R). \end{cases} \tag{1.4}$$