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Bifurcation of limit cycles from an n-dimensional linear center inside a class of piecewise linear differential systems

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ABSTRACT

Let n be an even integer. We study the bifurcation of limit cycles from the periodic orbits of the n-dimensional linear center given by the differential system

$$\dot{x}_1 = -x_2, \qquad \dot{x}_2 = x_1, \ \ldots, \ \dot{x}_{n-1} = -x_n, \qquad \dot{x}_n = x_{n-1},$$

perturbed inside a class of piecewise linear differential systems. Our main result shows that at most $(4n-6)^{n/2-1}$ limit cycles can bifurcate up to first-order expansion of the displacement function with respect to a small parameter. For proving this result we use the averaging theory in a form where the differentiability of the system is not needed.

1. Introduction and statement of the main result

Piecewise linear differential systems appear in a natural way in the control theory, and in the study of electrical circuits. These systems can present complicated dynamical phenomena such as those exhibited by general nonlinear differential systems. One of the main ingredients in the qualitative description of the dynamical behavior of a differential system is the number and the distribution of its limit cycles.

The goal of this paper is to study, in $\mathbb{R}^n$ for all n even, the existence of limit cycles of the control systems of the form

$$\dot{x} = A_0 x + \varepsilon F(x), \tag{1}$$

with $|\varepsilon| \neq 0$ a sufficiently small real parameter, where A_0 is equal to

$$A_0 = \begin{pmatrix} 0 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & -1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

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and $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is given by

$$F(x) = Ax + \varphi(k^T x)b,$$

with $A \in \mathcal{M}_n(\mathbb{R})$, $k, b \in \mathbb{R}^n \setminus \{0\}$ and $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ is the piecewise linear function

$$\varphi(x_1) = \begin{cases} -1 & \text{if } x_1 \in (-\infty, -1), \\ x_1 & \text{if } x_1 \in [-1, 1], \\ 1 & \text{if } x_1 \in (1, \infty), \end{cases} \tag{2}$$

where $x = (x_1, \ldots, x_n)^T$. The independent variable is denoted by t, vectors of $\mathbb{R}^n$ are column vectors, and k^T denotes a transposed vector.

For $\varepsilon = 0$ system (1) becomes

$$\dot{x}_1 = -x_2, \qquad \dot{x}_2 = x_1, \ldots, \dot{x}_{n-1} = -x_n, \qquad \dot{x}_n = x_{n-1}. \tag{3}$$

We note that the origin of (3) is a *global isochronous center*, i.e. all its orbits different from the origin are periodic with the same period 2π. A *limit cycle* of a differential system is an isolated periodic orbit in the set of all periodic orbits of the system. The *Poincaré map* (or equivalently, the *displacement map*) is a suitable tool for studying limit cycles of autonomous systems (detailed explanations can be found in [1] or [2]; see also Section 3). We recall that a limit cycle of a system corresponds to an isolated zero of the displacement function.

We must mention that there are many papers studying the bifurcation of limit cycles from the periodic orbits of a center, but almost all of them perturb a 2-dimensional center; see for instance the book [3] and the references quoted there. Of course, there are papers dedicated to perturb centers of dimension >2, but not too much. Later on we shall comment some of those papers.

The limit cycles of system (1) for $\varepsilon \neq 0$ sufficiently small that we shall study will be the ones bifurcating from the periodic orbits of the n-dimensional center (3) (i.e. of system (1) with $\varepsilon = 0$). As we shall see later on this study has been made for $n = 2, 4$. Here we shall do it for $n \geq 6$ even. Our main result can be stated as follows.

Theorem 1. *For all $n \geq 6$ even at most $(4n-6)^{n/2-1}$ limit cycles of system (1) with $\varepsilon \neq 0$ sufficiently small can bifurcate from the periodic orbits of the n-dimensional center (3), up to first-order expansion of the displacement function of system (1) with respect to the small parameter ε.*

Theorem 1 will be proved in Section 5.

In Section 2, we shall present the scheme of the proof of Theorem 1, mainly based on four lemmas.

We emphasize that the bifurcation from $\varepsilon = 0$ to $\varepsilon \neq 0$ in Theorem 1 takes place for $\varepsilon > 0$ and for $\varepsilon < 0$ sufficiently small, i.e. on both sides of the value $\varepsilon = 0$. We remark that in a Hopf bifurcation, the limit cycle only appears on one side of the bifurcation value of the parameter, but in our case in which the limit cycles bifurcate from periodic orbits of the period annulus of a center they appear on both sides of $\varepsilon = 0$.

The proof of Theorem 1 is based on the first-order averaging method. We will present this method in Section 3, in the form obtained in [4]. The advantage of this result is that the smoothness assumptions for the vector field of the differential system are minimal. In particular, it can be applied to piecewise linear differential systems, which are not C^2 (not even C^1), as it is required in its classical version; see for instance, Theorem 11.5 of [5]. This non-differential application of the averaging method to control systems was used for the first time in [6]. This method has been used frequently for computing periodic orbits; see for instance [7,8]. From the paper [9], we can study the stability of the limit cycles of Theorem 1; for more details see Remark 16.

Ref. [10] can be consulted for a theoretical discussion about suitable transformations of high dimensional differential systems which are small perturbations of a center, into the standard form for averaging. The general idea is to relate this change of variables to the first integrals of the unperturbed center.

We would like to add some comments related to our approach to the problem of counting the limit cycles of piecewise linear differential systems. We have chosen here to study bifurcation with respect to a small parameter from the periodic orbits of a center, up to first-order expansion of the displacement map. For some values of the coefficients, this order is sufficient for finding the exact number of limit cycles. But in some cases the first-order expansion of the displacement map can be identically zero, then a higher order averaging theory is needed. The study can be done by using second-, third-, …order averaging theory. A key point in these studies is the relation between the averaging theory and the displacement map due to the fact that the displacement map of a piecewise linear differential system is analytic in a neighborhood of a limit cycle.

2. Scheme of the proof of Theorem 1

First in Lemma 2 we shall reduce the n parameters of the vector b in the definition of the function $F(x)$ to one.