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Existence of mild solutions for abstract semilinear evolution equations in Banach spaces

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ABSTRACT

In this paper, we use a new fixed point theorem to study semilinear evolution equations with the initial conditions in Banach spaces. The results obtained here improve and generalize many known results.

1. Introduction

In this paper, we discuss the nonlocal initial value problem,

$$\begin{cases} x'(t) = Ax(t) + f(t, x(t)), \quad t \in I = [0, 1], \\ x(0) = g(x), \end{cases} \tag{1.1}$$

where A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators (i.e. C_0-semigroup) $T(t)$ in Banach space X, and $f : I \times X \to X, g : C([0, 1]; X) \to X$ are given X-valued functions.

The study of nonlocal initial value problems was initiated by Byszewski [1]. Since these represent mathematical models of various phenomena in Physics, Byszewski's work was followed by many others. For instance, Byszewski [1,2], Byszewski and Lakshmikantham [3] prove the existence and uniqueness of mild solutions and classical solutions when f and g satisfy Lipschitz-type conditions. Ntouyas and Tsamotas [4,5] study the case of compactness conditions of f and $T(t)$. In [6] Xue discusses the nonlocal initial value problem when f and g are compact. In [7] Fan give the existence of mild solutions of the nonlocal Cauchy problem when $T(t)$ is equicontinuous. The purpose of this paper is to continue the study of these authors. We prove the existence results of mild solutions for (1.1) without the compactness on $T(t)$ and f. So our work extends and improves many main results such as those in [6,7].

The organization of this work is as follows. In Section 2, we recall some definitions and facts about C_0 semigroup $T(t)$ and the measure of noncompactness. In Section 3, we give the existence of mild solutions of (1.1) under following conditions of g and $T(t)$: g is compact and $T(t)$ is equicontinuous.

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2. Preliminaries

Let $(X, \|\cdot\|)$ be a real Banach space. We denote by $C([0,1];X)$ the space of X-valued continuous functions on $[0,1]$ with the norm $\|x\| = \sup\{\|x(t)\|; t \in [0,1]\}$, and by $L^1(0,1;X)$ the space of X-valued Bochner functions on $[0,1]$ with the norm $\|x\| = \int_0^1 \|x(s)\| ds$. A C_0-semigroup $T(t)$ is said to be compact if $T(t)$ is compact for any $t > 0$. If the semigroup $T(t)$ is compact then $t \to T(t)x$ are equicontinuous at all $t > 0$ with respect to x in all bounded subsets of X; i.e. the semigroup $T(t)$ is equicontinuous. In this paper, we suppose that A generates a C_0 semigroup $T(t)$ on X. Since no confusion may occur, we denote by α the Hausdorff measure of noncompactness on both X and $C([0,1];X)$.

By a mild solution of the nonlocal initial value problem (1.1), we mean the function $x \in C([0,1];X)$ which satisfies

$$x(t) = T(t)g(x) + \int_0^t T(t-s)f(s,x(s))ds, \tag{2.1}$$

for all $t \in [0,1]$.

To prove the existence results in this paper we need the following lemmas.

Lemma 2.1 ([8]). *If $W \subseteq C([0,1];X)$ is bounded, then $\alpha(W(t)) \le \alpha(W)$ for all $t \in [0,1]$, where $W(t) = \{x(t); x \in W\} \subseteq X$. Furthermore if W is equicontinuous on $[0,1]$, then $\alpha(W(t))$ is continuous on $[0,1]$, and $\alpha(W) = \sup\{\alpha(W(t)); t \in [0,1]\}$.*

Lemma 2.2 ([9]). *If $\{u_n\}_{n=1}^{\infty} \subset L^1(0,1;X)$ is uniformly integrable, then $\alpha(\{u_n(t)\}_{n=1}^{\infty})$ is measurable and*

$$\alpha\left(\left\{\int_0^t u_n(s)ds\right\}_{n=1}^{\infty}\right) \le 2\int_0^t \alpha(\{u_n(s)\}_{n=1}^{\infty})ds. \tag{2.2}$$

Lemma 2.3 ([10]). *If the semigroup $T(t)$ is equicontinuous and $\eta \in L^1(0,1;\Re^+)$, then the set $\{t \to \int_0^t T(t-s)x(s)ds; x \in L^1(0,1;\Re^+), \|x(s)\| \le \eta(s),$ for a.e. $s \in [0,1]\}$ is equicontinuous on $[0,1]$.*

Lemma 2.4 ([11]). *If W is bounded, then for each $\varepsilon > 0$, there is a sequence $\{u_n\}_{n=1}^{\infty} \subseteq W$, such that*

$$\alpha(W) \le 2\alpha(\{u_n\}_{n=1}^{\infty}) + \varepsilon. \tag{2.3}$$

Lemma 2.5 ([12]). *Suppose that $0 < \epsilon < 1$, $h > 0$ and let*

$$S = \epsilon^n + C_n^1 \epsilon^{n-1} h + C_n^2 \epsilon^{n-2} \frac{(h)^2}{2!} + \cdots + \frac{(h)^n}{n!}, \quad n \in N^+.$$

Then $S = o(\frac{1}{n^s})(n \to +\infty)$, where $s > 1$ is an arbitrary real number.

Lemma 2.6 ([13] *Fixed Point Theorem*). *Let F be a closed and convex subset of a real Banach space X, let $A : F \to F$ be a continuous operator and $A(F)$ be bounded. For each bounded subset $B \subset F$, set*

$$A^1(B) = A(B), A^n(B) = A(\bar{co}(A^{n-1}(B))), \quad n = 2, 3, \ldots.$$

If there exist a constant $0 \le k < 1$ and a positive integer n_0 such that for each bounded subset $B \subset F$,

$$\alpha(A^{n_0}(B)) \le k\alpha(B),$$

then A has a fixed point in F.

3. Main results

In this section by using the Hausdorff measure of noncompactness in Banach space, we give the existence results of the nonlocal initial value problem (1.1). Here we list the following hypotheses.

(1) The C_0 semigroup $T(t)$ generated by A is equicontinuous. We denote $N = \sup\{\|T(t)\|; t \in [0,1]\}$.
(2) $g : C([0,1];X) \to X$ is continuous and compact, there exists positive constants c and d such that $\|g(x)\| \le c\|x\| + d$, $\forall x \in C([0,1];X)$.
(3) $f : [0,1] \times X \to X$ satisfies the Carathéodory type conditions, i.e. $f(\cdot, x)$ is measurable for all $x \in X$, and $f(t, \cdot)$ is continuous for a.e. $t \in [0,1]$.
(4) There exists a function $m \in L^1(0,1;\Re^+)$ and a nondecreasing continuous function $\Omega : \Re^+ \to \Re^+$ such that

$$\|f(t,x)\| \le m(t)\Omega(\|x\|),$$

for all $x \in X$, and a.e. $t \in [0,1]$.