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An identification problem for a time periodic nonlinear wave equation in non cylindrical domains

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ABSTRACT

The existence of a time-periodic solution of a free boundary nonlinear wave equation in non cylindrical domains is established. The problem arises in the study of the identification of the coefficient of the wave equation and of the boundary of the region from the observed values of the solution in a fixed subregion.

1. Introduction

The purpose of this paper is to study time-periodic solutions of a free boundary problem for a nonlinear wave equation. The problem arises in the identification of the coefficient of the wave equation and of the free boundary of the region from the observed values of the solution in a fixed subregion.

Let Ω be a bounded open subset of R^n with a smooth boundary. Consider the problem

$$\begin{aligned} & w'' - \nabla\{k(x)\nabla w\} + |w|^{p-2}w = f \quad \text{in } Q_\alpha,\ w = 0 \text{ on } \Gamma_\alpha, \\ & \{w, w'\}|_{t=0} = \{w, w'\}|_{t=T} \end{aligned} \tag{1.1}$$

where $2 \le p \le 2(n-1)/(n-2)$ and α is in the control set

$$\mathcal{A} = \left\{\alpha : \alpha \ge 0,\ \|\alpha\|_{C^{0,\lambda}(0,T;C_0^{1,\lambda}(\Omega))} \le 1, \alpha(.,0) = \alpha(.,T), \alpha' \le 0 \text{ for } t \in [0,T_*], \alpha' \ge 0 \text{ for } t \in [T_*,T]\right\}$$

while k is in the set of admissible coefficients

$$\mathcal{K} = \left\{k : k \ge k_0 > 0\ \forall x \in \Omega,\ \|k\|_{C^{0,\lambda}(\Omega)} \le 1\right\}.$$

We denote by $\Omega_\alpha^+(t)$ the set

$$\Omega_\alpha^+(t) = \left\{x : x \in \Omega, 0 < \alpha(x,t) < c, c \in (0, \inf_{\alpha\in\mathcal{A}} \max_Q \alpha(x,t)]\right\}$$

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with

$$Q_\alpha = \bigcup \Omega_\alpha^+(t) \times \{t\}, \qquad \Gamma_\alpha = \bigcup \partial\Omega_\alpha^+(t) \times \{t\}.$$

The existence of a time-periodic weak solution of (1.1) will be established in Section 3 for any f in a nonempty subset of $L^2(Q)$. Let $\{\chi, \Psi\}$ be two time-periodic functions in $L^2(0, T; L^2(G))$ with G being an open subset of $\Omega_\alpha^+(t)$ for all t and all $\alpha \in \mathcal{A}$. In Section 4, we consider the optimization problem

$$A = \inf\left\{\int_0^T \int_G \{|w - \chi| + |w' - \Psi|\}\mathrm{d}x\mathrm{d}t : \forall w \text{ solution of (1.1)}, \forall k \in \mathcal{K}, \ \forall\alpha \in \mathcal{A}\right\}. \tag{1.2}$$

Clearly, if $\{\chi, \Psi\}$ are the observed values of a time-periodic solution of (1.1) and of its time derivative in $G \times (0, T)$, then $A = 0$. It will be shown that

$$A = \int_0^T \int_G \{|\hat{w} - \chi| + |\hat{w}' - \Psi|\}\mathrm{d}x\mathrm{d}t$$

where $\hat{w}$ is a solution of (1.1) with $\{\hat{k}, \hat{\alpha}\} \in \mathcal{K} \times \mathcal{A}$. The determination of $\{\hat{k}, \hat{\alpha}, \hat{w}\}$ leads us to the study of the free boundary problem

$$w'' - \nabla\big\{k(w)\nabla w\big\} + |w|^{p-2}w = f \quad \text{in } Q_{\mathcal{P}w}, w = 0 \text{ on } \Gamma_{\mathcal{P}w},$$
$$\big\{w, w'\big\}|_{t=0} = \big\{w, w'\big\}|_{t=T} \tag{1.3}$$

where $\mathcal{P}$ is the projection of $L^2(Q)$ onto the compact set convex subset $\mathcal{A}$. It will be shown that $\hat{w}$ is a solution of (1.3).

In sharp contrast with the one-dimensional case, the literature on time-periodic solutions of the wave equation in higher dimension is scarce. The one-dimensional case was studied extensively by Berti–Bolle [1,2], Brezis–Nirenberg [3], Rabinowitz [4,5] and others. Pioneering work on time-periodic solutions for the linear wave equation in cylindrical domains was done by Glowinski and his collaborators in [6] and by Glowinski–Rossi [7]. Controllability and fictitious domains were used to treat the problem numerically. In [8], the author has shown the existence of time-periodic for the nonlinear wave equation in non cylindrical domains of R^3 and treated the transmission problem for the linear wave equation with a free time-periodic free interface. The existence of time-periodic solutions of the nonlinear wave equation in given cylindrical domains was established in [9]. The identification problem for time-periodic solutions of a nonlinear wave equation seems new.

2. Notations and preliminary results

Let $\mathcal{P}$ be the projection of $L^2(Q)$ onto the compact convex subset $\mathcal{A}$ i.e. for any y in $L^2(Q)$ there exists a unique $\mathcal{P}y$ in $\mathcal{A}$ such that

$$\|y - \mathcal{P}y\|_{L^2(Q)} = \inf\big\{\|y - \alpha\|_{L^2(Q)} : \forall\alpha \in \mathcal{A}\big\}.$$

It is known that $\mathcal{P}$ is a non expansive mapping in $L^2(Q)$ i.e.

$$\|\mathcal{P}y - \mathcal{P}z\|_{L^2(Q)} \le \|y - z\|_{L^2(Q)} \quad \forall y, z \in L^2(Q).$$

Let $\mathcal{K}$ be the compact convex subset of $L^2(\Omega)$ defined in the Introduction and let

$$\gamma(y) = \sup\left\{(k, y)_{L^2(Q)} - \|k\|^2_{L^2(\Omega)} : k \in \mathcal{K}\right\}. \tag{2.1}$$

Then γ is a l.s.c. convex function of $L^2(Q)$ into R and hence its sub-gradient $\partial\gamma(y)$ exists and is a set-valued mapping of $L^2(Q)$ into subsets of $L^2(Q)$.

Proposition 2.1. *There exists k_* in $\mathcal{K}$ such that*

$$\gamma(y) = (y, k_*(y))_{L^2(Q)} - \|k_*(y)\|^2_{L^2(\Omega)}$$

and $k_*(y) \in \partial\gamma(y)$.

Proof. (1) Let $\{k_n\}$ be a maximizing sequence of the optimization problem (2.1) with

$$\gamma(y) + n^{-1} \le (k_n, y)_{L^2(Q)} - \|k_n\|^2_{L^2(\Omega)}.$$

Since $k_n \in \mathcal{K}$, there exists a subsequence such that

$$k_n \to k_* \quad \text{in } C_0^{0,\nu}(\Omega), \ \nu < \lambda, k_* \in \mathcal{K}.$$

Hence

$$\gamma(y) = (k_*, y)_{L^2(Q)} - \|k_*\|^2_{L^2(\Omega)}.$$