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Modified scattering operator for the Hartree–Fock equation

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ABSTRACT

We study the scattering problem for the Hartree–Fock equation

$$i\partial_t u + \frac{1}{2}\Delta u = f(u), (t,x) \in \mathbf{R} \times \mathbf{R}^n, n \geq 2 \tag{HRF}$$

where $u = (u_1, \dots, u_N)^t$ is a $\mathbf{C}^N$ $(N \geq 2)$-valued unknown function and $f(u) = (f_1(u), \dots, f_N(u))^t$ denotes a nonlinear term whose jth-element is defined by

$$f_j(u) = \int_{\mathbf{R}^n} V(x-y) \sum_{k=1}^{N} \left\{ |u_k(y)|^2 u_j(x) - u_j(y)\bar{u}_k(y) u_k(x) \right\} \mathrm{d}y,$$

where $V(x) = \lambda |x|^{-1}$ $(\lambda \in \mathbf{R})$ is called a Coulomb potential. We show that if $\frac{1}{2} < \delta < \alpha$, then the modified scattering operator for the system (HRF) is well-defined from a neighborhood at the origin in the space $\mathbf{H}^{0,\alpha}$ to a neighborhood at the origin in the space $\mathbf{H}^{0,\delta}$, where $\mathbf{H}^{0,k} = \left\{ \phi \in \mathbf{L}^2; \left(1+|x|^2\right)^{\frac{k}{2}} \phi \in \mathbf{L}^2 \right\}$.

1. Introduction

In this paper, we study the scattering problem for the nonlinear Schrödinger equation with nonlocal interaction:

$$i\partial_t u + \frac{1}{2}\Delta u = f(u), \quad (t,x) \in \mathbf{R} \times \mathbf{R}^n, \tag{1.1}$$

where space dimension is $n \geq 2$, Δ denotes the Laplace operator in x, $u = (u_1, \dots, u_N)^t$ is a $\mathbf{C}^N$ $(N \geq 2)$-valued unknown function of (t,x) and $f(u)$ denotes a nonlinear term. The j-th element of $f(u) = (f_1(u), \dots, f_N(u))^t$ is defined by

$$f_j(u) = \int_{\mathbf{R}^n} V(x-y) \sum_{k=1}^{N} \left\{ |u_k(y)|^2 u_j(x) - u_j(y)\bar{u}_k(y) u_k(x) \right\} \mathrm{d}y, \tag{1.2}$$

where $V(x)$ is called a Coulomb potential given by

$$V(x) = \lambda |x|^{-1}, \quad \left(x \in \mathbf{R}^n \setminus \{0\}\right) \tag{1.3}$$

and λ is a non-zero real constant. The system (1.1) is called a time-dependent Hartree–Fock equation and appears in the quantum mechanics as an approximation to a Fermionic N-body system. Our aim is to show existence of the modified scattering operator for the system (1.1). To do so, we will improve domain and range of a modified wave operator obtained in Wada [1]. As for a modified inverse wave operator, we will use results obtained by Wada [1].

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We introduce an $N \times N$ matrix $F(u, v) = \{F_{ij}(u, v)\}_{1 \le i,j \le N}$ whose (i, j)-element is defined by

$$F_{ij}(u, v) = V * \left\{ \left(\sum_{k=1}^{N} u_k \bar{v}_k \right) \delta_{ij} - u_i \bar{v}_j \right\}, \tag{1.4}$$

where "$*$" denotes the convolution for space variables, δ_{ij} is Kronecker's delta i.e. $\delta_{ii} = 1$, $\delta_{ij} = 0\ (i \neq j)$. Furthermore we define an $N \times N$ matrix $F(u) = F(u, u)$ and then we can express nonlinear term $f(u)$ as

$$f(u) = F(u) u.$$

We note that $F(u)$ is an N-dimensional Hermitian matrix.

$\mathcal{F}\phi$ or $\hat{\phi}$ denotes the Fourier transform of ϕ defined by

$$\mathcal{F}\phi(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbf{R}^n} e^{-ix\cdot\xi} \phi(x)\, dx,$$

and the inverse Fourier transformation $\mathcal{F}^{-1}$ is given by

$$\mathcal{F}^{-1}\phi(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbf{R}^n} e^{ix\cdot\xi} \phi(\xi)\, d\xi.$$

For $m, k \in \mathbf{R}$, we introduce the weighted Sobolev spaces:

$$\mathbf{H}^{m,k} = \left\{ \phi \in \mathcal{S}'(\mathbf{R}^n);\ \|\phi\|_{\mathbf{H}^{m,k}} = \left\| \left(1 + |x|^2\right)^{k/2} (1 - \Delta)^{m/2} \phi \right\|_{\mathbf{L}^2} < \infty \right\}.$$

Let u_+ be a given final state. $A = A(t, \xi)$ is an $N \times N$ matrix-valued function and the solution of the Cauchy problem

$$i\partial_t A = t^{-1} F\left(A\hat{u}_+\right) A, \quad t \ge 1,\ \xi \in \mathbf{R}^n \tag{1.5}$$

$$A(1, \xi) = I_N, \quad \xi \in \mathbf{R}^n, \tag{1.6}$$

where I_N is the $N \times N$ unit matrix.

Our purpose can be formulated as follows. We assume that the final data

$$u_+ \in \mathbf{H}^{0,\alpha} \quad \text{with } \frac{1}{2} < \beta < \alpha < 1$$

and the norm $\|u_+\|_{\mathbf{H}^{0,\alpha}}$ is sufficiently small. Then we will find a unique global solution $u \in \mathbf{C}\left([0, \infty); \mathbf{H}^{0,\beta}\right)$ of (1.1) satisfying

$$\lim_{t \to +\infty} \left(u(t) - (it)^{-\frac{n}{2}} e^{\frac{i|x|^2}{2t}} A\left(t, \frac{x}{t}\right) \hat{u}_+\left(\frac{x}{t}\right) \right) = 0, \quad \text{in } \mathbf{H}^{0,\delta} \tag{1.7}$$

with $\frac{1}{2} < \delta < \beta$. This means that the modified wave operator for the system (1.1) is well-defined from a neighborhood at the origin in the space $\mathbf{H}^{0,\alpha}$ to a neighborhood at the origin in the space $\mathbf{H}^{0,\beta}$.

2. Several notations

Next we introduce several notations used in this paper.

For $\mathbf{C}^N$-valued functions, we denote the norm and the scalar product in $\mathbf{C}^N$ by $|\cdot|_{\mathbf{C}^N}$ and $(\cdot, \cdot)_{\mathbf{C}_N}$ respectively. For a $\mathbf{C}^N$-valued measurable function $\phi = (\phi_1, \ldots, \phi_N)^t$ on $\mathbf{R}^n$ and $1 \le p \le \infty$, $\phi \in \mathbf{L}^p$ means that $\phi_j \in \mathbf{L}^p$ for $j = 1, \ldots, N$ which is equivalent to $|\phi|_{\mathbf{C}^N} \in \mathbf{L}^p$. Its norm is defined by

$$\|\phi\|_{\mathbf{L}^p} \equiv \left\| |\phi(\cdot)|_{\mathbf{C}^N} \right\|_{\mathbf{L}^p}.$$

For $\mathbf{C}^N$-valued measurable functions $\phi = (\phi_1, \ldots, \phi_N)^t$ and $\psi = (\psi_1, \ldots, \psi_N)^t$ on $\mathbf{R}^n$, their $\mathbf{L}^2$-scalar product is defined by

$$(\phi, \psi)_{\mathbf{L}^2} \equiv \int_{\mathbf{R}^n} (\phi(x), \psi(x))_{\mathbf{C}^N}\, dx = \sum_{j=1}^{N} \left(\phi_j, \psi_j\right)_{\mathbf{L}^2}.$$

For matrix-valued functions, we introduce the following notations and function spaces. We denote by $\mathbf{M}_N$ the set of $N \times N$ matrices with complex elements. For $A = \left(a_{j,k}\right)_{1 \le j,k \le N} \in \mathbf{M}_N$, $|A|_{\mathbf{M}_N}$ denotes the operator norm of A in $\mathbf{C}^N$, that is

$$|A|_{\mathbf{M}_N} \equiv \sup_{|u|_{\mathbf{C}^N} \neq 0} \frac{|Au|_{\mathbf{C}^N}}{|u|_{\mathbf{C}^N}}.$$