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Analyticity of the Cauchy problem for two-component Hunter–Saxton systems

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ABSTRACT

This paper is mainly concerned with the periodic Cauchy problem for a generalized two-component μ-Hunter–Saxton system with analytic initial data. The analyticity of its solutions is proved in both variables, globally in space and locally in time. The obtained result can be also applied to its special cases—the classical integrable two-component Hunter–Saxton system, the generalized μ-Hunter–Saxton equation and the classical Hunter–Saxton equation.

1. Introduction

In this paper we mainly consider the periodic Cauchy problem of the following two-component μ-Hunter–Saxton system:

$$\begin{cases} u_{txx} = 2\mu(u)u_x - 2u_x u_{xx} - u u_{xxx} + k\rho\rho_x + \gamma_1 u_{xxx} + \gamma_2 \rho_{xx}, \\ \rho_t = (\rho u)_x + \gamma_2 u_{xx} + \gamma_3 \rho_x, \\ u(0,x) = u_0(x), \\ \rho(0,x) = \rho_0(x), \end{cases} \tag{1.1}$$

where $t \in \mathbb{R}$, $x \in \mathbb{T}$, $\mathbb{T} \equiv \mathbb{R}/\mathbb{Z}$, $\mu(u) \equiv \int_{\mathbb{T}} u(t,x)\mathrm{d}x$, $k = \pm 1$ and $\overrightarrow{\gamma} \equiv (\gamma_1, \gamma_2, \gamma_3) \in \mathbb{R}^3$.

System (1.1) was recently studied in [1] when we replace t by $-t$ and choose $k = 1$. The author proved that the two-component μ-Hunter–Saxton equation (2-μ-HS) is a bi-Hamiltonian Euler equation and can also be viewed as a bi-variational equation.

For $\mu(u) = 0$ and $\overrightarrow{\gamma} = \overrightarrow{0}$, system (1.1) becomes the classical Hunter–Saxton system (HS-system). It arises in the short-wave (or high-frequency) limit of the two-component Camassa–Holm system (2CH) [2,3] derived from the Green–Naghdi equations, which are approximations to the governing equations for water waves. The main motivation for seeking and studying such a system lies in capturing nonlinear phenomena such as wave-breaking and traveling waves which are not

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exhibited by small-amplitude models [4–6]. The HS-system is a particular case of the Gurevich–Zybin system describing the dynamics in a model of non-dissipative dark matter (see [7] and the references therein). For $\rho \not\equiv 0$, peakon solutions of the HS-system have been analyzed in [2]. Moreover, its Cauchy problem has been discussed in [8].

For $\rho \equiv 0$ and $\overrightarrow{\gamma} = \overrightarrow{0}$, system (1.1) reduces to a generalized Hunter–Saxton equation (μ-HS) lying between the Hunter–Saxton and Camassa–Holm equations, and which describes evolution of rotators in liquid crystals with an external magnetic field and self-interaction [9]. The authors proved that it is not only an Euler equation on the diffeomorphism group of the circle corresponding to a natural right-invariant Sobolev metric but also bi-Hamiltonian and admits both cusped and smooth traveling wave solutions which are natural candidates for solitons. Furthermore, the term $2\mu(u)u_x$ has a strong effect on well-posedness of its Cauchy problem in that it is responsible for μ-HS admitting blow-up solutions and global solutions in time [9].

For $\rho \equiv 0$, $\overrightarrow{\gamma} = \overrightarrow{0}$ and $\mu(u) = 0$, system (1.1) becomes the classical Hunter–Saxton equation (HS) [10] modeling the propagation of weakly nonlinear orientation waves in a massive nematic liquid crystal. In the Hunter–Saxton equation [10], x is the space variable in a reference frame moving with the linearized wave velocity, t is a slow-time variable and $u(t, x)$ is a measure of the average orientation of the medium locally around x at time t. More precisely, the orientation of the molecules is described by the field of unit vectors $(\cos u(t, x), \sin u(t, x))$ [11]. The single-component model also arises in a different physical context as the high-frequency limit [12,13] of the Camassa–Holm equation for shallow water waves [14,15] and a re-expression of the geodesic flow on the diffeomorphism group of the circle [16] with a bi-Hamiltonian structure [17] which is completely integrable [18]. The Hunter–Saxton equation also has a bi-Hamiltonian structure [15,19] and is completely integrable [13,20]. The initial value problem for the Hunter–Saxton equation on the line (nonperiodic case) was studied by Hunter and Saxton in [10]. Using the method of characteristics, they showed that smooth solutions exist locally and break down in finite time; see [10]. The occurrence of blow-up can be interpreted physically as the phenomenon by which waves that propagate away from the perturbation knock the director field out of its unperturbed state [10]. The initial value problem for the Hunter–Saxton equation on the circle $\mathbb{T}$ was discussed in [11]. The author proved the local existence of strong solutions to the periodic Hunter–Saxton equation, showed that all strong solutions except space-independent solutions blow up in finite time by using Kato semigroup method [21]. Moreover, the behavior of the solutions exhibits different features.

The analyticity of solutions to Euler equations of hydrodynamics has been studied extensively. (It was initiated by [22,23] and later further developed in [24–28] and in the papers of [29,30] where the approach is based on a contraction type argument in a suitable scale of Banach spaces.) In particular, the analyticity of the Cauchy problem for two-component shallow water systems has been proved in [31]. However, in this paper we will prove the analyticity of solutions to system (1.1) in both variables, with x on the circle $\mathbb{T}$ and t in a neighborhood of zero, provided that the initial data is analytic on $\mathbb{T}$. Note that the classical Cauchy–Kowalevski theorem does not apply to (1.1) since the initial line $t = 0$ is characteristic. Thus the result above can be viewed as an extended Cauchy–Kowalevski theorem for the nonlinear case (1.1). More precisely, we have the following main theorem.

Theorem 1.1. *Let* $\binom{u_0}{\rho_0}$ *be real analytic on* $\mathbb{T}$. *There exists an* $\varepsilon > 0$ *and a unique solution* $\binom{u}{\rho}$ *of the Cauchy problem* (1.1) *that is analytic on* $(-\varepsilon, \varepsilon) \times \mathbb{T}$.

2. Proof of our main theorem

First, we will rewrite the initial value problem (1.1) into a nonlocal form. Integrating both sides of the first equation of (1.1) w.r.t. x, we obtain

$$u_{tx} = 2\mu(u)u - (uu_x)_x + \frac{1}{2}u_x^2 + \frac{k}{2}\rho^2 + \gamma_1 u_{xx} + \gamma_2 \rho_x + a(t),$$

where $a(\cdot) : \mathbb{R} \longrightarrow \mathbb{R}$ is an arbitrary continuous function, denoted by $a(t) \in C(\mathbb{R})$.

Integrating once more in x, we have

$$u_t = \gamma_1 u_x - uu_x + \gamma_2 \rho + \partial_x^{-1}\left(2\mu(u)u + \frac{1}{2}u_x^2 + \frac{k}{2}\rho^2 + a(t)\right) + b(t).$$

That is

$$\partial_t u = \partial_x\left(\gamma_1 u - \frac{1}{2}u^2\right) + \partial_x^{-1}\left(2\mu(u)u + \frac{1}{2}(\partial_x u)^2 + \frac{k}{2}\rho^2 + a(t)\right) + \gamma_2\rho + b(t),$$

where $\partial_x^{-1} f(x) \equiv \int_0^x f(y)\mathrm{d}y$ and $b(t) \in C(\mathbb{R})$.

On the other hand, the second equation of (1.1) is equivalent to the following equation:

$$\partial_t \rho = \partial_x(\rho u + \gamma_2(\partial_x u) + \gamma_3 \rho).$$