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Generalized convexity and the existence of finite time blow-up solutions for an evolutionary problem

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ABSTRACT

In this paper we study a class of nonlinearities for which a nonlocal parabolic equation with Neumann–Robin boundary conditions, for p-Laplacian, has finite time blow-up solutions.

1. Introduction

It is a well known fact that convexity plays an important role in the different parts of mathematics, including the study of boundary value problems. The aim of our paper is to introduce a new class of generalized convex functions and to illustrate its usefulness in establishing a sufficient condition for the existence of finite time blow-up solutions for the evolutionary problem

$$\begin{cases} u_t - \Delta_p u = f(|u|) - \dfrac{1}{m(\Omega)} \displaystyle\int_\Omega f(|u|)\mathrm{d}x & \text{in } \Omega \\ |\nabla u|^{p-2} \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

with the initial conditions

$$u(x, 0) = u_0(x) \quad \text{on } \Omega, \text{ where } \int_\Omega u_0 \mathrm{d}x = 0. \tag{1.2}$$

Here $\Omega \subset \mathbb{R}^N$ is a bounded regular domain of class $C^2, f : [0, \infty) \mapsto [0, \infty)$ is a locally Lipschitz function, $m(\Omega)$ represents the Lebesgue measure of the domain Ω, and $\Delta_p = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$, for $p \geq 2$, is the p-Laplacian operator.

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The particular case where $p = 2$ was recently considered by Soufi et al. [1], and Jazar and Kiwan [2] (under the assumption that f is a power function of the form $f(u) = u^\alpha$, with $\alpha > 1$), and also by the present authors [3] (for f belonging to a larger class of nonlinearities).

The problems of type (1.1) and (1.2) arise naturally in mechanics, biology and population dynamics. See [4–8]. For example, if we consider a couple or a mixture of two equations of the above type, the resulting problem describes the temperatures of two substances which constitute a combustible mixture, or represents a model for the behavior of densities of two diffusion biological species which interact each other.

2. Generalized convexity of order α

According to the classical Hermite–Hadamard inequality, the mean value of a continuous convex function $f : [a, b] \to \mathbb{R}$ lies between the value of f at the midpoint of the interval $[a, b]$ and the arithmetic mean of the values of f at the endpoints of this interval, that is,

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \le \frac{f(a)+f(b)}{2}. \tag{HH}$$

Moreover, each side of this double inequality characterizes convexity in the sense that a real-valued continuous function f defined on an interval I is convex if its restriction to each compact subinterval $[a, b] \subset I$ verifies the left hand side of (HH) (equivalently, the right hand side of (HH)). See [9,10] for details.

In what follows we will be interested in a class of generalized convex functions motivated by the right hand side of the Hermite–Hadamard inequality.

Definition 1. A real-valued function f defined on an interval $[a, \infty)$ belongs to the class GC_α (for some $\alpha > 0$), if it is continuous, nonnegative, and

$$\frac{1}{\alpha+1} f(t) \ge \frac{1}{t-a}\int_a^t f(x)\mathrm{d}x \quad \text{for } t \text{ large enough.} \tag{2.1}$$

Using calculus, one can see easily that the condition (2.1) is equivalent to the fact that the ratio

$$\frac{\frac{1}{t-a}\int_a^t f(x)\mathrm{d}x}{(t-a)^\alpha} \tag{2.2}$$

is nondecreasing for t bigger than a suitable value $A \ge a$. In turn, this implies that the mean value $\frac{1}{t-a}\int_a^t f(x)\mathrm{d}x$ has a polynomial growth at infinity.

According to the Hermite–Hadamard inequality, every nonnegative, continuous and convex function $f : [a, \infty) \to \mathbb{R}$ with $f(a) = 0$ belongs to the class GC_1. The converse is not true because the membership of a function $f : [a, \infty) \to \mathbb{R}$ to the class GC_α yields only an asymptotic inequality of the form

$$\frac{1}{\alpha+1} f(t) + \frac{\alpha}{\alpha+1} f(a) \ge \frac{1}{t-a}\int_a^t f(x)\mathrm{d}x \quad \text{for } t \text{ large enough.}$$

If $g \in C^1([0, \infty))$ and g is nondecreasing, then the function $f(x) = g(x)(x-a)^\alpha$ belongs to the class CG_α $([0, \infty))$, whenever $\alpha > 0$. In fact,

$$\begin{aligned}\frac{1}{t-a}\int_a^t f(x)\mathrm{d}x &= \frac{(t-a)^\alpha}{\alpha+1} g(t) - \frac{1}{t-a}\int_a^t g'(x)\frac{(x-a)^{\alpha+1}}{\alpha+1}\mathrm{d}x \\ &\le \frac{1}{\alpha+1} f(t).\end{aligned}$$

As a consequence, $(x + \sin x)x$ provides an example of function of class GC_1 on $[0, \infty)$ which is not convex.

No positive constant can be a function of class GC_α for any $\alpha > 0$.

Also, the restriction of a function $f : [a, \infty) \to \mathbb{R}$ of class GC_α to a subinterval $[b, \infty)$ is not necessarily a function of class GC_α.

In the sequel we will describe some other classes of functions of class GC_α.

The following concept of generalized convexity is due to Varosanec [11] and generalizes the usual convexity, s-convexity, the Godunova–Levin functions and P-functions.

Definition 2. Suppose that $h : [0, 1] \to \mathbb{R}$ is a function such that $h(\lambda) + h(1-\lambda) \ge 1$ for all $\lambda \in [0, 1]$. A nonnegative function f defined on an interval I is called h-convex if

$$f(\lambda x + (1-\lambda)y) \le h(\lambda)f(x) + h(1-\lambda)f(y) \tag{2.3}$$

whenever $\lambda \in [0, 1]$, and $x, y \in I$.