Nonlinear Analysis 75 (2012) 433–443

Contents lists available at SciVerse ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Exponential problem on a compact Riemannian manifold without boundary

Liang Zhao *

School of Mathematical Sciences, Beijing Normal University, Beijing 100875, PR China
Laboratory of Mathematics and Complex Systems, Ministry of Education, Beijing 100875, PR China

ARTICLE INFO

Article history:
Received 25 October 2010
Accepted 21 August 2011
Communicated by Ravi Agarwal

MSC:
58J05
58E30
35J60
35B33
35J20

Keywords:
Trudinger–Moser inequality
Adams–Fontana inequality
Exponential growth

ABSTRACT

Let (M, g) be an n-dimensional compact Riemannian manifold without boundary. A Trudinger–Moser-type inequality says that

$$\sup_{\|u\|_{W^{1,n}}\le 1}\int_M e^{\alpha_n|u|^{\frac{n}{n-1}}}\,dv_g<\infty,$$

where $\|u\|_{W^{1,n}}$ is the usual Sobolev norm of $u\in W^{1,n}(M)$, $\alpha_n=n\omega_{n-1}^{\frac{1}{n-1}}$, and ω_{n-1} is the area of the unit sphere $\mathbb{S}^{n-1}$. Using this inequality, when $\varepsilon>0$ is small enough, we establish sufficient conditions under which the quasilinear equation

$$-\Delta_n u+|u|^{n-2}u=f(x,u)+\varepsilon h(x)$$

has at least two nontrivial weak solutions in $W^{1,n}(M)$, where $-\Delta_n u=-\mathrm{div}_g(|\nabla u|^{n-2}\nabla u)$, $f(x,u)$ behaves like $e^{\gamma|u|^{\frac{n}{n-1}}}$ as $|u|\to\infty$ for some $\gamma>0$, and $h\not\equiv 0$ belongs to the dual space of $W^{1,n}(M)$.

© 2011 Elsevier Ltd. All rights reserved.

1. Introduction and main results

Let (M, g) be a compact Riemannian manifold of dimension n $(n\ge 2)$ without boundary and $W^{1,n}(M)$ be the usual Sobolev space. Here $W^{1,n}(M)$ is the completion of $C^\infty(M)$ under the norm

$$\|u\|_{W^{1,n}}=\left(\int_M(|\nabla u|^n+|u|^n)dv_g\right)^{\frac{1}{n}},\tag{1.1}$$

where ∇ is the gradient operator and dv_g is the volume element of (M, g). A special case of the Fontana inequalities (see [1]) says that

$$\sup_{\int_M u\,dv_g=0,\ \|\nabla u\|_{L^n}\le 1}\int_M e^{\alpha_n|u|^{\frac{n}{n-1}}}dv_g<\infty,\tag{1.2}$$

where $\|\cdot\|_{L^n}$ denotes the $L^n(M)$ norm, $\alpha_n=n\omega_{n-1}^{\frac{1}{n-1}}$, and ω_{n-1} is the area of the unit sphere $\mathbb{S}^{n-1}$. If α_n is replaced by any larger number, the integrals in (1.2) are still finite, but cannot be bounded uniformly by any constant. Inequality (1.2) is a

* Correspondence to: School of Mathematical Sciences, Beijing Normal University, Beijing 100875, PR China. Tel.: +86 13141282291; fax: +86 10 58808202.
E-mail address: liangzhao@bnu.edu.cn.

0362-546X/$ – see front matter © 2011 Elsevier Ltd. All rights reserved.
doi:10.1016/j.na.2011.08.043

manifold case of the well-known Trudinger–Moser inequalities (see [2–5]). In [6], replacing the hypothesis $\int_M u dv_g = 0$ and $\int_M |\nabla u|^n dv_g \le 1$ by $\int_M (|\nabla u|^n + |u|^n) dv_g \le 1$, do Ó and Yang proved that (1.2) is still valid. Namely, they proved the following theorem.

Theorem A *([6], Theorem 1.2). Let (M, g) be a compact Riemannian manifold of dimension n without boundary. Then*

$$\sup_{u\in W^{1,n}(M),\ \|u\|_{W^{1,n}}\le 1} \int_M e^{\alpha_n |u|^{\frac{n}{n-1}}} dv_g < \infty, \tag{1.3}$$

where $\alpha_n = n\omega_{n-1}^{\frac{1}{n-1}}$ and ω_{n-1} is the area of the unit sphere $\mathbb{S}^{n-1}$. Furthermore, this inequality is sharp: when α_n is replaced by any larger number, the integrals in (1.3) are still finite, but the supremum is infinity.

An elementary proof of Theorem A is based on (1.2) and the Young inequality (see [6,7]). A similar idea together with a rearrangement argument can be applied to the Trudinger–Moser inequality in the whole space $\mathbb{R}^n$ (see [8]). Another proof is based on the blow-up analysis, and is thus much more complicated (see [9]). The modified Fontana inequality (1.3) will be more natural when we consider related partial differential equations on manifolds. In addition to the modified Fontana inequality (1.3), the following manifold version of Lion's inequality (see [10]) is another key ingredient in our argument.

Theorem B. *Let $\{u_k\}$ be a sequence in $W^{1,n}(M)$ such that $\|u_k\|_{W^{1,n}} = 1$, $u_k \rightharpoonup u$ in $W^{1,n}(M)$, $u_k \to u$ in $L^n(M)$, and $\nabla u_k(x) \to \nabla u(x)$ for almost every $x \in M$. Then, for any $p < (1 - \|u\|^n_{W^{1,n}})^{-\frac{1}{n-1}}$,*

$$\sup_k \int_M e^{p\alpha_n |u_k|^{\frac{n}{n-1}}} dv_g < +\infty.$$

The proof of Theorem B is based on the Brézis–Lieb lemma (see [11]). Such kinds of theorem are very important when studying exponential problems (see [12,13,7]).

As applications of the two theorems above, we study the existence result of the following quasilinear equation:

$$-\Delta_n u + |u|^{n-2}u = f(x, u) + \varepsilon h(x) \quad \text{in } M, \tag{1.4}$$

where $-\Delta_n u = -\mathrm{div}_g(|\nabla u|^{n-2}\nabla u)$; the nonlinearity $f(x, u)$ has the maximal growth on u which allows us to treat problem (1.4) variationally in the Sobolev space $W^{1,n}(M)$. For the Euclidean case, similar problems have been studied extensively (see [14–20] and the references therein). To present our existence result, we assume that f satisfies the following hypotheses.

(H_1) $f : M \times \mathbb{R} \to \mathbb{R}$ is continuous and there exist constants $C > 0$ and $\beta > 0$ such that

$$|f(x, s)| \le Ce^{\beta|s|^{\frac{n}{n-1}}}.$$

(H_2) There exist constants $R > 0$ and $A > 0$ such that, for all $s \ge R$ and all $x \in M$,

$$0 < F(x, s) = \int_0^s f(x, t)dt \le Af(x, s).$$

(H_3) $f(x, s) \ge 0$ for all $(x, s) \in M \times [0, \infty)$ and $f(x, 0) = 0$ for all $x \in M$.

(H_4) $\limsup_{s\to 0^+} \frac{nF(x,s)}{s^n} < 1$ uniformly for $x \in M$.

(H_5) There exists $\alpha_0 > 0$ such that the following limit holds uniformly for all $x \in M$:

$$\lim_{s\to+\infty} sf(x, s)e^{-\alpha_0 s^{\frac{n}{n-1}}} = +\infty.$$

Our main result is the following.

Theorem 1.1. *Assume (H_1) –(H_5). Then there exists $\varepsilon_1 > 0$ such that, for each $0 < \varepsilon < \varepsilon_1$, Eq. (1.4) has at least two nontrivial solutions.*

Solutions to Eq. (1.4) are critical points of the functional

$$J_\varepsilon(u) := \frac{1}{n}\int_M (|\nabla u|^n + |u|^n)dv_g - \int_M F(x, u)dv_g - \int_M \varepsilon h(x)u dv_g,$$

where $F(x, s) = \int_0^s f(x, t)dt$ for all $x \in M$ and $s \in \mathbb{R}$. In view of the structure of J_ε, particularly its first term $\int_M (|\nabla u|^n + |u|^n)dv_g$, it is reasonable to use Theorem A instead of Fontana's original inequality (1.2) to study the compactness of the Palais–Smale sequence of J_ε. The existence of the second solution of (1.4) is based on the mountain-pass theory. A similar idea has been used by de Figueiredo et al. (see [16]) to establish the same result in the case when (M, g) is replaced by any smooth bounded domain in $\mathbb{R}^2$.

The remaining part of this paper is organized as follows. In Section 2, we prove Theorem B and study the geometric and variational structures of the functional J_ε. Then we prove Theorem 1.1 in Section 3.

Download English Version:

https://daneshyari.com/en/article/840367

Download Persian Version:

https://daneshyari.com/article/840367

Daneshyari.com