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# Solution profiles beyond quenching for a radially symmetric multi-dimensional parabolic problem

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ABSTRACT

Let $T \le \infty$, $R > 0$, $\Omega = (0, R) \times (0, T)$, $\chi\{S\}$ be the characteristic function of the set $S$. This article studies the following parabolic problem:

$$r^{N-1}u_t - \left(r^{N-1}u_r\right)_r = r^{N-1}f(u)\,\chi(\{u < c\}) \quad \text{in } \Omega,$$
$$u(r, 0) = 0 \quad \text{on } [0, R], \qquad u_r(0, t) = 0 = u(R, t) \quad \text{for } 0 < t < T,$$

where $N$ is a positive integer, $f$ is a given twice continuously differentiable function on $[0, c)$ for some constant $c$ with $f(0) > 0, f' > 0, f'' \ge 0$, and $\lim_{u \to c^-} f(u) = \infty$. It is shown that under some additional conditions on $f$, the problem has a weak solution, and all weak solutions of the problem tend to a unique steady-state (nonclassical) solution $U(r)$ as $t$ tends to infinity. Furthermore, increasing the length $R$ increases the interval where $U(r) \equiv c$ by the same amount.



## 1. Introduction

Let $x = (x_1, x_2, x_3, \ldots, x_N)$ be a point in the $N$-dimensional Euclidean space $\mathbb{R}^N$, $Hu = u_t - \Delta u$, $D = \{x : |x| < R\}$ for some positive constant $R$, $\partial D$ be its boundary, $\bar{D}$ be its closure, $T \le \infty$, and

$$\chi(S) = \begin{cases} 1 & \text{if } u \in S, \\ 0 & \text{if } u \notin S, \end{cases}$$

be the characteristic function of the set $S$. Our main purpose here is to study what happens beyond quenching through the radial solutions of the following problem:

$$\left.\begin{array}{l} Hu = f(u)\,\chi(\{u < c\}) \quad \text{in } D \times (0, T), \\ u(x, 0) = 0 \quad \text{on } \bar{D}, \qquad u(x, t) = 0 \quad \text{for } x \in \partial D, 0 < t < T, \end{array}\right\} \tag{1.1}$$

where $f(u)$ is twice continuously differentiable on $[0, c)$ for some constant $c$ such that $f(0) > 0, f' > 0, f'' \ge 0$, $\lim_{u \to c^-} f(u) = \infty$, $\int_0^c f(u)\,du = K_0$ for some positive constant $K_0$, and for some positive constants $K_1$, $K_2$ and $K_3$,

$$f'(u)\left(\frac{c - u}{f(u)}\right)^2 \le K_1, \tag{1.2}$$

$$\int_u^c f(s)\,ds \le \min\left\{K_2(c - u)f(u), K_3(c - u)^\gamma\right\}, \tag{1.3}$$

where $\gamma$ is a constant between 0 and 2.

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For any constant $\epsilon > 0$, let

$$f_\epsilon(u) = f(u)\frac{c-u}{\epsilon f(u)+c-u}.$$

Then, $\lim_{\epsilon\to 0} f_\epsilon(u) = f(u)\,\chi(\{u<c\})$, and $f_\epsilon(u) < f(u)$. We have

$$f'_\epsilon(u) = \frac{-\epsilon f^2(u) + (c-u)^2 f'(u)}{(\epsilon f(u)+c-u)^2} \le \frac{(c-u)^2 f'(u)}{(\epsilon f(u)+c-u)^2} \le \frac{1}{\epsilon^2} f'(u)\left(\frac{c-u}{f(u)}\right)^2.$$

Using (1.2), we obtain $f'_\epsilon(u) \le K_1/\epsilon^2$. Let us consider the regularized problem,

$$\left.\begin{array}{l} Hu^\epsilon = f_\epsilon(u^\epsilon) \quad \text{in } D\times(0,T), \\ u^\epsilon(x,0)=0 \quad \text{on } \bar{D}, \qquad u^\epsilon(x,t)=0 \quad \text{for } x\in\partial D,\ 0<t<T \end{array}\right\}. \tag{1.4}$$

Since

$$H0 = 0 \le f_\epsilon(0) \quad \text{in } D\times(0,T), \qquad u^\epsilon(x,0)=0 \quad \text{on } \bar{D}, \qquad u^\epsilon(x,t)=0 \quad \text{for } x\in\partial D,\ 0<t<T,$$

it follows that 0 is a lower solution of the problem (1.4). Because

$$f_\epsilon(c) = \lim_{u^\epsilon\to c^-} f_\epsilon(u^\epsilon) = \frac{\lim_{u^\epsilon\to c^-}(c-u^\epsilon)}{\epsilon + \left(\lim_{u^\epsilon\to c^-}(c-u^\epsilon)\right)\left(\lim_{u^\epsilon\to c^-}\frac{1}{f(u^\epsilon)}\right)} = 0,$$

we have

$$\begin{array}{l} Hc = 0 = f_\epsilon(c) \quad \text{in } D\times(0,T), \\ c \ge u^\epsilon(x,0) = 0 \quad \text{on } \bar{D}, \qquad c\ge u^\epsilon(x,t)=0 \quad \text{for } x\in\partial D,\ 0<t<T. \end{array}$$

Thus, $c$ is an upper solution.

The following existence result follows from Theorem 4.2.2 of Ladde et al. [1, p. 143].

**Lemma 1.1.** *The problem (1.4) has a solution $u^\epsilon \in C^{2+\alpha,1+\alpha/2}\left(\overline{D\times(0,T)}\right)$, where $0<\alpha<1$.*

A proof similar to that of Lemma 1 of Chan and Kaper [2] gives the following result.

**Lemma 1.2.** *The problem (1.4) has at most one solution. In $D\times(0,T)$, $0<u^\epsilon<c$, and $u^\epsilon$ is a strictly increasing function of $t$.*

Let $u$ denote $\lim_{\epsilon\to 0} u^\epsilon$ if the limit exists.

**Lemma 1.3.** *If $0<\epsilon_1<\epsilon_2$, then $u^{\epsilon_1} > u^{\epsilon_2}$ in $D\times(0,T)$. Furthermore, $u(x,t)$ is continuous on $\overline{D\times(0,T)}$.*

**Proof.** The proof of $u^\epsilon$ being a strictly decreasing function of $\epsilon$ is similar to that of Lemma 3 of Chan and Kong [3]. Since $0<u^\epsilon<c$ in $D\times(0,T)$, and $u^\epsilon \in C^{2,1}\left(\overline{D\times(0,T)}\right)$ is strictly increasing as $\epsilon$ decreases, it follows from the Dini Theorem (cf. Stromberg [4, p. 143]) that $u^\epsilon$ converges uniformly on $\overline{D\times(0,T)}$, and hence, $u(x,t)$ is continuous on $\overline{D\times(0,T)}$. □

**Theorem 1.1.** *In $\{u<c\}\cap(D\times(0,T))$, the limit $u(x,t)$ satisfies $Hu=f(u)$ in the classical sense.*

**Proof.** By Lemma 1.1, the problem (1.4) has a solution $u^\epsilon \in C^{2+\alpha,1+\alpha/2}\left(\overline{D\times(0,T)}\right)$. For any $(x_0,t_0)\in\{(x,t): u(x,t)<c\}\cap(D\times(0,T))$, we have $u(x_0,t_0)<c$. From the proof of Lemma 1.3, $u^\epsilon$ converges uniformly to $u$ as $\epsilon$ decreases. There exists a constant $m$ depending on $u(x_0,t_0)$ and not on $\epsilon$ such that $0<u^\epsilon\le m<c$ in some neighborhood $\Sigma$ of $(x_0,t_0)$. Since for $u\le m$,

$$|f-f_\epsilon| = f(u)\left(1-\frac{c-u}{\epsilon f(u)+c-u}\right) = \frac{\epsilon f^2(u)}{\epsilon f(u)+c-u} \le \frac{\epsilon f^2(m)}{\epsilon f(u)+c-u} \le \frac{\epsilon f^2(m)}{c-u} \le \frac{\epsilon f^2(m)}{c-m},$$

we have that $f_\epsilon \to f$ uniformly as $\epsilon\to 0$ on $\{(x,t): u(x,t)\le m\}$. Let $p$ be a constant such that $p>(N+2)/(2-\alpha)$. Because $u^\epsilon(x,t)\le m$ for $(x,t)\in\Sigma$, we have $\|u^\epsilon\|_{L^p(\Sigma)} = \left(\int_\Sigma (u^\epsilon)^p\,dxdt\right)^{1/p} \le m\left(\int_\Sigma dxdt\right)^{1/p} \le k_1$ for some positive constant $k_1$. Since $f_\epsilon(u^\epsilon) < f(u^\epsilon)$ for any $\epsilon>0$, and $f$ is increasing, we have

$$\|f_\epsilon(u^\epsilon)\|_{L^p(\Sigma)} \le \|f(u^\epsilon)\|_{L^p(\Sigma)} \le \|f(m)\|_{L^p(\Sigma)}.$$