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The existence of solutions to nonlinear second order periodic boundary value problems

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ABSTRACT

We consider the existence of periodic boundary value problems of nonlinear second order ordinary differential equations. Under certain half Lipschitzian type conditions several existence results are obtained. As applications positive periodic solutions of some ϕ-Laplacian type equations and Duffing type equations are investigated.

1. Introduction

The lower and the upper solution technique has been extensively investigated in studying boundary value problems of nonlinear differential equations. The ideal dates back to the work of Perron [1] on the Dirichlet problem for harmonic functions. The theory for treating boundary value problems of second order nonlinear ordinary differential equations was established in the late 1960s by Jackson [2]. Significant contributions were also made by Schmitt in [3–5], and Jackson and Schrader in [6]. Some notable applications were given in [7,8] by the author who characterized the existence of a positive solution to the Dirichlet problem of a type of sublinear differential equations. Related results may also be seen in [9,10].

In this paper we consider the general periodic boundary value problem

$$\begin{cases} -x'' = f(t, x, x') & (t \in [0, 1]) \\ x(0) = x(1), & x'(0) = x'(1), \end{cases} \tag{1.1}$$

where $f \in C\left([0, 1] \times \mathbb{R}^2, \mathbb{R}\right)$, the space of all real-valued continuous functions on $[0, 1] \times \mathbb{R}^2$. Classical results concerning this problem by using lower and upper solutions are due to Schmitt [3], who investigated the existence of a solution under a Nagumo condition. Recent relevant results may be found in [11,12]. On the other hand, periodic solutions of special type of equations like Duffing equations have been attracting great attention lately (see [13–15]).

In this paper we start with a sort of half Lipschitzian type conditions to establish an existence theorem. Then we will consider some one-sided y-growth conditions on the function $f(t, x, y)$ to obtain some more existence results. As applications we will consider the forced pendulum equations with curvature, that is the ϕ-Laplacian type equations of the form

$$\left(\frac{x'}{\sqrt{1 + x'^2}}\right)' + \mu(t) \sin x - \ell(t, x)x' = e(t),$$

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and Duffing type equations of the form

$$x'' + p(t)g(x) - q(t)h(x') = e(t),$$

deriving several existence results about positive periodic solutions of the equations.

2. Upper and lower solutions

Let $\alpha(t), \beta(t) \in C^2[0, 1]$. We call $\alpha(t)$ a *lower solution* and $\beta(t)$ an upper solution of problem (1.1) if

$$-\alpha''(t) \le f(t, \alpha, \alpha') \quad (t \in [0, 1]),\ \alpha(0) = \alpha(1);$$
$$-\beta''(t) \ge f(t, \beta, \beta') \quad (t \in [0, 1]),\ \beta(0) = \beta(1).$$

Given $a > 0$ and $b \in \mathbb{R}$ there is a unique solution $h(t)$ to the linear problem

$$\begin{cases} -h'' = -ah + bh' & (t \in [0, 1]) \\ h(1) - h(0) = 0, & h'(1) - h'(0) = 1. \end{cases} \tag{2.1}$$

Precisely

$$h(t) = \frac{(1 - e^{\lambda_2})e^{\lambda_1 t} + (e^{\lambda_1} - 1)e^{\lambda_2 t}}{(\lambda_1 - \lambda_2)(e^{\lambda_1} - 1)(1 - e^{\lambda_2})},$$

where λ_1, λ_2 are respectively the positive and negative roots of the equation $\lambda^2 + b\lambda - a = 0$, that is

$$\lambda_1 = \frac{-b + \sqrt{b^2 + 4a}}{2} \quad \text{and} \quad \lambda_2 = \frac{-b - \sqrt{b^2 + 4a}}{2}.$$

Clearly, $h(t) > 0$ for $t \in [0, 1]$ and $h'(0) < 0$.

Lemma 2.1. *Let $\alpha(t)$ and $\beta(t)$ be, respectively, lower and upper solutions of problem (1.1). Let $r_1 = \alpha'(0) - \alpha'(1)$ and $r_2 = \beta'(1) - \beta'(0)$. Suppose that*

(E_0) *there are constants $a, b, \delta \in \mathbb{R}$ with $a > 0$ such that*

$$f(t, \beta, \beta') - f(t, \alpha, \alpha') \ge -a(\beta - \alpha) + b(\beta' - \alpha') - \delta \quad (t \in [0, 1]).$$

Then

$$(\beta(t) - \alpha(t)) - (r_1 + r_2)h(t) + \delta/a \ge 0 \quad t \in [0, 1].$$

Proof. Let

$$\theta(t) = (\beta(t) - \alpha(t)) - (r_1 + r_2)h(t) + \delta/a.$$

Then $\theta(0) = \theta(1)$ and $\theta'(0) = \theta'(1)$. Moreover

$$\begin{aligned} \theta'' &\le -\big(f(t, \beta, \beta') - f(t, \alpha, \alpha')\big) - (r_1 + r_2)h''(t) \\ &\le a\,((\beta - \alpha) - (r_1 + r_2)h) - b\left((\beta' - \alpha') - (r_1 + r_2)h'\right) + \delta \\ &= a\theta(t) - b\theta'(t) \end{aligned} \tag{2.2}$$

for $t \in [0, 1]$. If the conclusion of the lemma were not true, then there would exist $t_1 \in [0, 1)$ such that $\theta(t_1) < 0$, $\theta'(t_1) = 0$, and $\theta''(t_1) \ge 0$. But on the other hand, if this is the case, by inequality (2.2) we would derive $\theta''(t_1) \le a\theta(t_1) < 0$. This is a contradiction. Thus $\theta(t) \ge 0$ for all $t \in [0.1]$. □

Lemma 2.2. *Suppose that condition (E_0) holds for some $a, b, \delta \in \mathbb{R}$ with $a > 0$. Let*

$$k_0 = -\frac{h'(0)}{h(0)} = -\frac{\lambda_1(1 - e^{\lambda_2}) + \lambda_2(e^{\lambda_1} - 1)}{(e^{\lambda_1} - e^{\lambda_2})}.$$

Then $k_0 > 0$ and

$$(\beta' - \alpha') + k_0(\beta - \alpha) - (r_1 + r_2)(k_0 h + h') + k_0\delta/a \ge 0 \quad t \in [0, 1]. \tag{2.3}$$