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Existence and regularity of minimizers of a functional for unsupervised multiphase segmentation

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ABSTRACT

We consider a variational model for image segmentation proposed in Sandberg et al. (2010) [12]. In such a model the image domain is partitioned into a finite collection of subsets denoted as phases. The segmentation is unsupervised, i.e., the model finds automatically an optimal number of phases, which are not required to be connected subsets. Unsupervised segmentation is obtained by minimizing a functional of the Mumford–Shah type (Mumford and Shah, 1989 [1]), but modifying the geometric part of the Mumford–Shah energy with the introduction of a suitable scale term. The results of computer experiments discussed in [12] show that the resulting variational model has several properties which are relevant for applications. In this paper we investigate the theoretical properties of the model. We study the existence of minimizers of the corresponding functional, first looking for a weak solution in a class of phases constituted by sets of finite perimeter. Then we find various regularity properties of such minimizers, particularly we study the structure of triple junctions by determining their optimal angles.

1. Introduction

The segmentation problem in image analysis consists in looking for a decomposition of an image into homogeneous regions corresponding to meaningful parts of objects. In recent years a number of variational models have been proposed for the segmentation problem. Mumford and Shah [1] proposed to minimize the functional

$$\mathcal{E}_{ms}(u,\Gamma)=\alpha\mathcal{H}^1(\Gamma)+\beta\int_{\Omega\setminus\Gamma}|\nabla u|^2dx+\int_{\Omega}|u-u_o|^2dx, \tag{1}$$

where $\Omega\subset\mathbb{R}^2$ is a bounded Lipschitz image domain, $u_o:\Omega\longrightarrow\mathbb{R}_+\cup\{0\}$ is a bounded function representing the given image, $\mathcal{H}^1$ is the 1-dimensional Hausdorff measure, and α, β are positive weights. The functional has to be minimized over all closed sets $\Gamma\subset\overline{\Omega}$ and all $u\in\mathcal{C}^1(\Omega\setminus\Gamma)$. The function u represents a piecewise smooth (i.e., denoised) approximation of the input image u_o, and the set Γ represents the union of boundaries of the regions constituting the segmentation. If Γ is regular enough, then the measure $\mathcal{H}^1(\Gamma)$ is simply the total length of the boundaries.

In [2,3] Chan and Vese introduced a multiphase variational model for image segmentation based on the Mumford–Shah functional and the level set method. They considered both a piecewise constant and a piecewise smooth approximation of the image u_o. In the piecewise constant case the variational model looks for a local minimizer of the functional

$$\mathcal{E}_{cv}(u,\Gamma)=\alpha\mathcal{H}^1(\Gamma)+\sum_{i=1}^{K}\int_{\chi_i}|u-u_o|^2dx, \tag{2}$$

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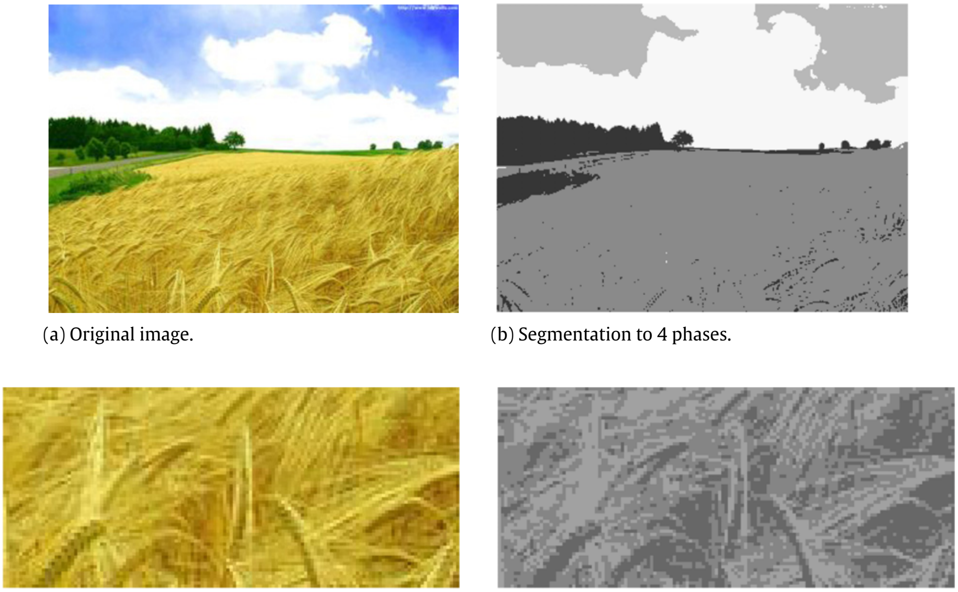

(a) Original image. (b) Segmentation to 4 phases.

(c) Original (zoom). (d) Segmentation to 3 phases.

Fig. 1. The original image (a) is automatically segmented to four phases in image (b), each gray color represents different phases of the segmentation result. When zoomed into the field area and (c) is given as an original image, image (c) is automatically further segmented to three phases in image (d).
Source: Figures from Ref. [12].

where K is a given integer, χ_i, $i = 1, \ldots, K$, are open subsets that constitute a Borel partition of the image domain Ω, here Γ is the union of the part of the boundaries of the χ_i inside Ω, so that

$$\Gamma = \bigcup_{i=1}^{K} \partial\chi_i \cap \Omega, \quad \Omega = \Gamma \bigcup_{i=1}^{K} \chi_i, \tag{3}$$

and the function u is constant on every subset χ_i. It is easy to see that, for a fixed Γ, the functional $\mathcal{E}_{cv}$ is minimized with respect to the function u by setting, for each χ_i, u equal to the mean value of u_o in χ_i. The subsets χ_i are denoted as phases and are not required to be connected. The functional $\mathcal{E}_{cv}$ is a piecewise constant version of the Mumford–Shah functional, since $\nabla u(x) = 0$ in $\Omega \setminus \Gamma$. With a level-set formulation and implementation, the model is frequently referred to as the Chan–Vese model for either two-phase ($K = 2$), or multiphase ($K > 2$), segmentation.

Extending this idea, there is a number of region-based multiphase segmentation models introduced, such as [4–11] for $K \geq 2$. However, except for the case of two-phase segmentation, the multiphase case can have some sensitivity issues. Typically the number of phases $K > 2$ is pre-determined and result can depend on the initial guess used in the local minimization of the functional.

The model proposed in [12] addresses these issues, that the model automatically chooses a reasonable number of phases K, as it segments the image via the minimum of the following functional:

$$\mathcal{E}(K, \chi_1, \ldots, \chi_K) = \mu \left(\sum_{i=1}^{K} \frac{P(\chi_i)}{|\chi_i|} \right) \mathcal{H}^1(\Gamma) + \sum_{i=1}^{K} \int_{\chi_i} |u_o - c_i|^2 dx. \tag{4}$$

Here $P(\chi_i)$ denotes the perimeter of a phase χ_i, $|\chi_i|$ denotes the 2-dimensional area of a phase χ_i, Γ is defined by (3), for any $i = 1, \ldots, K$, c_i is the mean value of u_o in χ_i, and μ is a positive parameter. Notice that both K and the χ_is are all unknown variables.

Compared to the piecewise constant Mumford–Shah model, one difference is the newly added weight $\sum_i \frac{P(\chi_i)}{|\chi_i|}$ in front of the length term. The ratio $\frac{P(\chi_i)}{|\chi_i|}$, called scale term, is related to Cheeger sets, which are widely studied in the Calculus of Variations [13,14]. The Cheeger problem consists of finding a single subset of Ω minimizing the ratio perimeter/area, while in our problem this new weight is the summation of the scale terms for multiple phases. Such a weight gives an effective property of the model, allowing an unsupervised multiphase segmentation, as it has been discussed in [12]. Here, unsupervised segmentation means the automatic selection of the optimal number of phases K by functional minimization. The number of phases recovered in an optimal segmentation can be controlled by means of the parameter μ: large values of μ favor fewer phases with larger areas, while small values of μ prefer more phases with smaller area (see [12] for further details). Fig. 1 shows an example from [12], where K is automatically selected depending on the focus of the image. In this example the value $\mu = 1$ has been used. This model has many properties in addition to being an automatic segmentation,