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Lipschitz maps and primitives for continuous functions in quasi-Banach spaces

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ABSTRACT

We show that for a wide class of non-locally convex quasi-Banach spaces X that includes the spaces ℓ_p for $0 < p < 1$, there exists a continuous function $f : [0, 1] \to X$ failing to have a primitive, thus solving a problem raised by M.M. Popov in 1994. We also construct the first known examples of functions in $\mathcal{C}^{(1)}([a, b], X)$ that fail to be Lipschitz.

1. Introduction

If X is a Banach space, every continuous map $f : [a, b] \to X$ is Riemann-integrable and the corresponding integral function, $F(t) = \int_a^t f(u)\,du$ is differentiable at every $t \in [a, b]$ with derivative $F'(t) = f(t)$, that is, F is a primitive of f. However, when X is a non-locally convex F-space, a classical theorem of Mazur and Orlicz [1] informs us about the existence of continuous X-valued functions on $[a, b]$ failing to be integrable. Popov investigated in [2] the properties of the Riemann integral for functions $f : [a, b] \to X$ where X is an F-space and showed that while some usual properties of this integral remain true in the non-locally convex setting, other properties and techniques, like the usual way of getting primitives for integrable functions, may be false. His work naturally led to the question whether every continuous function $f : [a, b] \to X$ has a primitive. Kalton provided an affirmative answer for the quasi-Banach spaces X which, like the L_p spaces for $p < 1$, have trivial dual [3], but the main question remained unsolved. In the first part of this paper we solve Popov's problem by showing that if the space ℓ_p with $0 < p < 1$ embeds isomorphically in a quasi-Banach space X with separating dual, then there exists an integrable continuous function $f : [0, 1] \to X$ failing to have a primitive. This will follow as a consequence of our main theorem.

Theorem 1.1 (*Main Theorem*). *Suppose $0 < p < 1$. Then there exists a continuous Riemann-integrable function $f : [0, 1] \to \ell_p$ whose integral function $F : [0, 1] \to \ell_p,\ t \to \int_0^t f(s)\,ds$ verifies:*

(a) *F is Lipschitz, i.e., there is $C > 0$ so that $\|F(s) - F(t)\|_p \le C|s - t|$ for all $s, t \in [0, 1]$;*
(b) *F is differentiable at every $t \in [0, 1)$ with derivative $F'(t) = f(t)$;*
(c) *F fails to have left derivative at $t = 1$.*

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In Section 4 we exploit the construction used in Section 2 in the proof of Theorem 1.1 to show that, unlike for Banach spaces, every non-locally convex quasi-Banach space X with separating dual admits a continuously differentiable function $f : [a, b] \to X$ which is not Lipschitz. Finally, Sections 5 and 6 gather remarks on the general problem of classifying those quasi-Banach spaces X for which every continuous function $f : [a, b] \to X$ has a primitive. We refer the reader to [2,3] for background and to [4,5] for the needed terminology and notation on quasi-Banach spaces.

2. Proof of the main theorem

The proof of Theorem 1.1 relies on the following construction inspired by [2]. Let $\tau = (t_k)_{k=1}^{\infty}$ be an increasing sequence of scalars contained in $(0, 1)$ tending to 1. With $t_0 = 0$, let us denote the interval $[t_{k-1}, t_k)$ by I_k and its length by λ_k, i.e., $\lambda_k = |I_k| = t_k - t_{k-1}$. This way we can write $[0, 1) = \cup_{k=1}^{\infty} I_k$ (disjoint union). For each $k \in \mathbb{N}$ let $f_{I_k} : [0, 1] \to \mathbb{R}$ be the nonnegative piecewise linear function supported on the interval I_k having a node at the midpoint of the interval $c_k = (t_k + t_{k-1})/2$ with $f_{I_k}(c_k) = 2$ and $f_{I_k}(t_{k-1}) = f_{I_k}(t_k) = 0$, i.e.,

$$f_{I_k}(t) = \begin{cases} \dfrac{4}{t_k - t_{k-1}}(t - t_{k-1}) & \text{if } t \in [t_{k-1}, c_k) \\ \dfrac{4}{t_k - t_{k-1}}(t - t_k) & \text{if } t \in [c_k, t_k) \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathbf{x} = (x_n)_{n=1}^{\infty}$ be a sequence of vectors in a quasi-Banach space X. We define the function $f = f(\tau, \mathbf{x}) : [0, 1] \to X$ as

$$f(t) = \begin{cases} f_{I_k}(t)x_k & \text{if } t \in I_k, \\ 0 & \text{if } t = 1. \end{cases} \tag{2.1}$$

Note that f is continuous and Riemann-integrable on $[0, 1)$ since for each $s < 1$ the set $f([0, s])$ is a finite-dimensional subspace of X. Let $F = F(\tau, \mathbf{x})$ be the corresponding integral function on $[0, 1)$,

$$F(t) = \int_0^t f(u)du. \tag{2.2}$$

The additivity of the Riemann-integral with respect to the interval gives that for $t \in I_n$,

$$F(t) = \sum_{k=1}^{n-1} \lambda_k x_k + \int_{t_{n-1}}^{t} f(u)\,du = \sum_{k=1}^{n} \lambda_k x_k - \int_t^{t_n} f(u)\,du. \tag{2.3}$$

Again, since $F([0, s])$ maps into a finite-dimensional subspace of X for each $s < 1$, F is differentiable with derivative $F'(t) = f(t)$ at every $t \in [0, 1)$. The next proposition deals mainly with the behavior of the functions f and F at the point $t = 1$ depending on the choice of $(\tau, \mathbf{x})$.

Proposition 2.1. *Let X be a quasi-Banach space. For a given pair $(\tau, \mathbf{x})$ we have the following.*

(i) *The function $f = f(\tau, \mathbf{x}) : [0, 1] \to X$ is continuous at 1, hence continuous on $[0, 1]$, if and only if $x_k \to 0$.*
(ii) *Suppose that X is p-convex for some $0 < p \le 1$. If (x_k) is bounded and the sequence (λ_k) verifies $\sum_{k=1}^{\infty} \lambda_k^p < \infty$, then f is Riemann-integrable on $[0, 1]$.*
(iii) *$F = F(\tau, \mathbf{x})$ can be extended continuously to $[0, 1]$ by putting $F(1) = \sum_{k=1}^{\infty} \lambda_k x_k$ if and only if the series $\sum_{k=1}^{\infty} \lambda_k x_k$ converges in X.*
(iv) *Suppose (x_k) is bounded. Then $F : [0, 1) \to X$ is Lipschitz if and only if there is $\mathsf{K} > 0$ so that for all integers m, n with $m < n$,*

$$\frac{\left\| \sum_{m+1 \le k \le n} \lambda_k x_k \right\|}{\sum_{m+1 \le k \le n} \lambda_k} = \frac{\left\| \sum_{m+1 \le k \le n} \lambda_k x_k \right\|}{t_n - t_m} \le \mathsf{K}. \tag{2.4}$$

In this case F extends to a Lipschitz function on the whole interval $[0, 1]$.
(v) *Suppose $x_k \to 0$. Then $F : [0, 1] \to X$ is differentiable with zero left-derivative at $t = 1$ if and only if*

$$\lim_{n\to\infty} \frac{\left\| \sum_{k \ge n+1} \lambda_k x_k \right\|}{\sum_{k \ge n+1} \lambda_k} = \lim_{n\to\infty} \frac{\left\| \sum_{k \ge n+1} \lambda_k x_k \right\|}{1 - t_n} = 0. \tag{2.5}$$