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A nondegenerate fuzzy optimality condition for constrained optimization problems without qualification conditions

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ABSTRACT

Fuzzy optimization conditions in terms of the Fréchet subdifferential for reflexive spaces were investigated by Borwein, Treiman and Zhu (1998) in [1]. To achieve the nondegenerate form, it is well known that some qualification conditions should be assumed. In this paper, we are going to prove that the nondegenerate fuzzy optimality condition even holds with no qualification conditions in Asplund spaces (in particular, reflexive spaces) for optimization problems with semi-continuous and continuous data. The results are even new in finite-dimensional frameworks.

1. Introduction

The paper is mainly devoted to the study of optimization problems with inequality and equality constraints as follows:

$$\begin{cases} \text{minimize } \varphi_0(x) \text{ subject to} \\ \varphi_k(x) \le 0, \quad k = 1, \dots, m, \\ \varphi_k(x) = 0, \quad k = m+1, \dots, m+r, \\ x \in \Omega, \end{cases} \tag{1.1}$$

where $\varphi_k : X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$, $k = 0, \dots, m+r$ are extended-real-valued functions on Asplund space X (a Banach space where every separable subspace has a separable dual) and Ω is a closed set in X. Assume that the functions φ_k are lower semi-continuous (l.s.c.) for $k = 0, \dots, m$ and that φ_k are continuous for $k = m+1, \dots, m+r$ around the point in question. When the space X is reflexive and $\bar{x}$ is an optimal solution of problem (1.1), Borwein, Treiman and Zhu in their elegant paper [1] established a very strong result, the so-called "fuzzy multiplier rule" to program (1.1). Briefly it says that for any $\varepsilon > 0$ and any weak* neighborhood V^* of 0 in X^*, there exist $x_k \in \mathbb{B}_\varepsilon(\bar{x})$ for $k = 0, \dots, m+r$, $\widehat{x} \in \mathbb{B}_\varepsilon(\bar{x}) \cap \Omega$, and $(\lambda_0, \dots, \lambda_{m+r}) \in \mathbb{R}_+^{m+r+1}$ with $\sum_{k=0}^{m+r} \lambda_k = 1$ such that

$$0 \in \lambda_0 \widehat{\partial} \varphi_0(x_0) + \sum_{k=1}^{m} \lambda_k \widehat{\partial} \varphi_k(x_k) + \sum_{k=m+1}^{m+r} \lambda_k \big(\widehat{\partial} \varphi_k(x_k) \cup \widehat{\partial}(-\varphi_k)(x_k)\big) + \widehat{N}(\widehat{x}; \Omega) + V^*, \tag{1.2}$$

where the notations $\widehat{\partial}$ and $\widehat{N}$ represent the *Fréchet subdifferentials* and *Fréchet normal cones* defined in Section 2. The result was extended to Asplund spaces independently by using different approaches in [2–4]. We say that the multiplier

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$(\lambda_0, \ldots, \lambda_{m+r})$ is *degenerate* if the obtained coefficient λ_0 is equal or close to zero. In fact, such multipliers do not provide much useful information to the original problem (1.1) since it may absorb the cost function φ_0. In optimization theory, one of the most important tasks is to find constraint qualifications which guarantee that the multiplier $(\lambda_0, \ldots, \lambda_{m+r})$ is not degenerate. Such conditions for "fuzzy" optimality conditions are known quite well in the literature; see, e.g., [5, Theorems 3.3.7 and 3.3.13]. Roughly speaking, under some conditions, for any $\varepsilon > 0$ and any weak* neighborhood V^* of 0 in X^*, there exist $x_k \in \mathbb{B}_\varepsilon(\bar{x})$ for $k = 0, \ldots, m + r$, $\widehat{x} \in \mathbb{B}_\varepsilon(\bar{x}) \cap \Omega$, and $(\lambda_1, \ldots, \lambda_{m+r}) \in \mathbb{R}^{m+r}_+$ such that

$$0 \in \widehat{\partial}\varphi_0(x_0) + \sum_{k=1}^{m} \lambda_k \widehat{\partial}\varphi_k(x_k) + \sum_{k=m+1}^{m+r} \lambda_k\big(\widehat{\partial}\varphi_k(x_k) \cup \widehat{\partial}(-\varphi_k)(x_k)\big) + \widehat{N}(\widehat{x}; \Omega) + V^*. \tag{1.3}$$

This inclusion is called the *nondegenerate* optimality condition of problem (1.1). From the appearances of [1,5] till now, it has been believed that inclusion (1.3) should be attached with some constraint qualifications (CQs). Our paper is going to show a significantly stronger conclusion: inclusion (1.3) can stand itself without any CQs. Following the main stream of the paper [1], some "fuzzy" estimates to each *Fréchet normal* of level sets and sublevel sets are established in the sequel. However, instead of using small dual balls of X^* to approximate the Fréchet normal cone as in [1,3,4], we utilize a weak* neighborhood of the origin in X^*. The remarkable advantage of this approach is that there is no qualification condition assumed. In fact, such estimates have been considered by Zhu in [4] for the *β-viscosity normal cone* in the *β-smooth Banach space* but those ones still need some restrictive conditions. By combining these results with the well-known "weak fuzzy sum rule" [6, Theorem 2], we easily prove inclusion (1.3).

The rest of the paper is organized as follows. In Section 2, we present some preliminaries in nonsmooth analysis and generalized differentiation. Two most important tools in our approach, the *weak fuzzy sum rule* and the *approximate extremal principle*, are also recalled here for convenience. Section 3 is devoted to the study of nondegenerate optimality conditions. The main result of the paper, Theorem 3.1 which ensures that (1.3) holds with no qualification condition in the Asplund space is stated here. To achieve the "exact" necessary optimality condition in terms of *limiting subdifferentials* and *singular subdifferentials*, we introduce a new condition, which is satisfied automatically when the set of all "fuzzy multipliers" in (1.3) is bounded with small ε given.

In Section 4, we present the proof of Theorem 3.1 in detail. We construct three technical lemmas whose proofs just follow from known calculus in variational analysis. Lemmas 4.2 and 4.3 are self-contained and seem to be nice refinements to some known results in [1,5,3,4]. The idea of using *nonsmooth infinite programming* in the proof of Lemma 4.2 is new in the literature. It somehow changes a program with finite constraints to a new one with infinite constraints. Then the techniques to derive necessary optimality conditions of infinite programs in [7, Theorem 3.18] and [8, Theorem 3.1] may help us to observe the former problem from a different viewpoint. At the end of this section we derive the proof of our main result by only combining these lemmata with the weak fuzzy sum rule.

Our notation and terminology are basically standard and can be found in [5,9]. For any Banach space X we denote by $\|\cdot\|$ its norm and X^* its topological dual with canonical pairing $\langle\cdot,\cdot\rangle$. The symbol $\xrightarrow{w^*}$ indicates the convergence in the weak* topology of X^*. For any $x \in X$ and $r > 0$, denote by $\mathbb{B}_r(x)$ the closed ball centered at x with radius r while $\mathbb{B}$ and $\mathbb{B}^*$ stand for the closed unit ball in the space and the dual space in question.

Given a set $\Omega \subset X$ and a function $\varphi : X \to \overline{\mathbb{R}}$, the symbols $x \xrightarrow{\Omega} \bar{x}$ and $x \xrightarrow{\varphi} \bar{x}$ mean that $x \to \bar{x}$ with $x \in \Omega$ and $x \to \bar{x}$ with $\varphi(x) \to \varphi(\bar{x})$ respectively. Finally in this section, for any set-valued mapping $F: Z \rightrightarrows X^*$, recall that the symbol

$$\operatorname*{Lim\,sup}_{z\to\bar{z}} F(z) := \left\{x^* \in X^* \mid \exists z_n \to \bar{z},\ \exists x_n^* \xrightarrow{w^*} x^* \text{ with } x_n^* \in F(z_n),\ n \in \mathbb{N}\right\} \tag{1.4}$$

signifies for the *sequential* Painlevé–Kuratowski outer/upper limit of F as $z \to \bar{z}$ with respect to the norm topology of Banach space Z and the weak* topology of X^*, where $\mathbb{N} := \{1, 2, \ldots\}$.

2. Preliminaries

Let X be Banach spaces and Ω be a subset of X. We define the *ε-normals* to Ω at $\bar{x} \in \Omega$ by

$$\widehat{N}_\varepsilon(\bar{x}; \Omega) := \left\{x^* \in X^* \mid \limsup_{x\xrightarrow{\Omega}\bar{x}} \frac{\langle x^*, x - \bar{x}\rangle}{\|x - \bar{x}\|} \le \varepsilon\right\}, \quad \varepsilon \ge 0. \tag{2.1}$$

When $\varepsilon = 0$, elements of (2.1) are called *regular/Fréchet normals* and their collection denoted by $\widehat{N}(\bar{x}; \Omega)$ is the *regular/Fréchet normal cone* to Ω at $\bar{x}$.

Now we define the *basic/limiting/Mordukhovich normal cone* to Ω at $\bar{x}$ via the sequential Painlevé–Kuratowski outer limit (1.4) as follows:

$$N(\bar{x}; \Omega) := \operatorname*{Lim\,sup}_{\substack{x\xrightarrow{\Omega}\bar{x}\\ \varepsilon\downarrow 0}} \widehat{N}_\varepsilon(x; \Omega). \tag{2.2}$$