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Hardy spaces associated with magnetic Schrödinger operators on strongly Lipschitz domains

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ABSTRACT

Let $p \in (0, 1]$, Ω be a strongly Lipschitz domain in $\mathbb{R}^n$ and $A := -(\nabla - i\vec{a}) \cdot (\nabla - i\vec{a}) + V$ a magnetic Schrödinger operator on $L^2(\Omega)$ satisfying the Dirichlet boundary condition, where $\vec{a} := (a_1, \ldots, a_n) \in L^2_{\rm loc}(\Omega, \mathbb{R}^n)$ and $0 \le V \in L^1_{\rm loc}(\Omega)$. In this paper, the authors introduce the Hardy space $H^p_A(\Omega)$ by the Lusin area function associated with A and establish its equivalent characterization via the non-tangential maximal function associated with $\{e^{-t\sqrt{A}}\}_{t>0}$. As applications, the authors obtain the boundedness of the Riesz transforms $L_k A^{-\frac{1}{2}}$, $k \in \{1, \ldots, n\}$, from $H^p_A(\Omega)$ to $L^p(\Omega)$ for $p \in (0, 1]$ and the fractional integral $A^{-\gamma}$ from $H^p_A(\Omega)$ to $H^q_A(\Omega)$ for $0 < p < q \le 1$ and $\gamma := \frac{n}{2}(\frac{1}{p} - \frac{1}{q})$, where L_k is the closure of $\frac{\partial}{\partial x_k} - ia_k$ in $L^2(\Omega)$.

1. Introduction

The theory of Hardy spaces on the n-dimensional Euclidean space $\mathbb{R}^n$, initiated by Stein and Weiss [1], plays an important role in various fields of analysis and has been transformed into a rich and multifaceted theory; see, [2,3]. It is known that the classical Hardy spaces on $\mathbb{R}^n$ are essentially related to the Laplacian $\triangle := \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ and good substitutes of Lebesgue spaces when $p \in (0, 1]$ in the study of the boundedness of operators.

One important aspect of the development in the theory of Hardy spaces is the study of Hardy spaces on domains Ω of $\mathbb{R}^n$; see, for example, [4–7]. Especially, let Ω be a strongly Lipschitz domain in $\mathbb{R}^n$ and $H^1_r(\Omega)$ the restriction to Ω of the Hardy space $H^1(\mathbb{R}^n)$. Under the so-called Dirichlet boundary condition, Auscher and Russ [8] proved that the space $H^1_r(\Omega)$ can be characterized by the non-tangential maximal function and the area integral function associated with $\{e^{-t\sqrt{L}}\}_{t>0}$, where L is an elliptic second-order divergence operator such that, for all $t \in (0, \infty)$, the kernel of e^{-tL} satisfies the Gaussian upper bound and the Hölder continuity. Recently, when $p \in (n/(n+1), 1]$, V is a nonnegative polynomial on $\mathbb{R}^n$ and $L := -\triangle + V$ a Schrödinger operator, Huang [9] introduced the Hardy space $H^p_{L,\,r}(\Omega)$ by restricting the Hardy space $H^p_L(\mathbb{R}^n)$, which was introduced and studied by Dziubański [10], to Ω and further established the characterizations of $H^p_{L,\,r}(\Omega)$ in terms of the non-tangential maximal function and the area integral associated with $\{e^{-t\sqrt{L}}\}_{t>0}$.

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In recent years, the study of the magnetic Schrödinger operator attracts a lot of attention; see, for example, [11–15]. Let $C_c^\infty(\Omega)$ be the *set of* $C^\infty(\Omega)$ *functions with compact support in* Ω and

$$A := \sum_{k=1}^{n} L_k^* L_k + V$$

the magnetic Schrödinger operator, where L_k is the *closure in* $L^2(\Omega)$ *of* $\frac{\partial}{\partial x_k} - ia_k$ *with domain* $C_c^\infty(\Omega)$, L_k^* the *adjoint operator of* L_k *in* $L^2(\Omega)$, $k \in \{1, \ldots, n\}$, $\vec{a} := (a_1, \ldots, a_n) \in L^2_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ the magnetic potential and $0 \le V \in L^1_{\mathrm{loc}}(\Omega)$ the electrical potential. When $\Omega := \mathbb{R}^n$, Shen [11] obtained the $L^p(\mathbb{R}^n)$, with $p \in (1, \infty)$, estimates and the weak-type (1, 1) estimate for the Riesz transforms $\{L_j L_k A^{-1}\}_{j,k=1}^n$ under slightly strong assumptions on $\vec{a}$ and V; Duong et al. [13] further established the boundedness of the *Riesz transforms* $\{L_k A^{-\frac{1}{2}}\}_{k=1}^n$ on $L^p(\mathbb{R}^n)$, with $p \in (1, 2]$, and the boundedness from the Hardy space $H^1_A(\mathbb{R}^n)$, introduced by Auscher et al. in [16], to $L^1(\mathbb{R}^n)$. Let $\mathcal{X}$ be a metric measure space and L a nonnegative self-adjoint operator satisfying the so-called Davies–Gaffney estimate. Hofmann et al. [17] introduced and characterized the Hardy space $H^1_L(\mathcal{X})$ in terms of atoms, molecules and the Lusin area function associated with the semigroup $\{e^{-t\sqrt{L}}\}_{t>0}$. These characterizations were, in [17], applied to the Schrödinger operator A on $\mathbb{R}^n$ with $\vec{a} := 0$ to establish the equivalent characterizations of $H^1_A(\mathbb{R}^n)$ in terms of the non-tangential maximal function $N_h f$ and the radial maximal function $R_h f$ associated with $\{e^{-t^2 A}\}_{t>0}$, and the non-tangential maximal function $N_P f$ and the radial maximal function $R_P f$ associated with $\{e^{-t\sqrt{A}}\}_{t>0}$, respectively. All these results were further generalized to Orlicz–Hardy spaces in [18], which include the Hardy spaces $H^p_A(\mathbb{R}^n)$ for $\vec{a} := 0$ and $p \in (0, 1]$ as a special case. In [19], the equivalent characterizations of $H^p_A(\mathbb{R}^n)$ with $p \in (0, 1]$ were established in terms of $N_h f$, $N_P f$, $R_h f$ and $R_P f$, respectively.

The purpose of this paper is to characterize Hardy spaces associated with the magnetic Schrödinger operator A on a strongly Lipschitz domain Ω. By Hundertmark and Simon [12, Theorem 3.3], one can see that, for all $t \in (0, \infty)$, the Poisson kernel $p_t(x, y)$ of $e^{-t\sqrt{A}}$ satisfies the Poisson upper bound on Ω. However, to the best of our knowledge, it is unclear whether $p_t(x, y)$ has the Hölder continuity on space variables or not. Without the regularity of p_t on space variables and, moreover, noticing that the atoms and the molecules of $H^p_A(\mathbb{R}^n)$ in [17,18] are closely connected to A, it seems that it is not convenient to introduce a useful Hardy space $H^p_A(\Omega)$ just by restricting the elements of $H^p_A(\mathbb{R}^n)$ in [17,18] to Ω as in [8] (see also [9]), since the regularity of the Poisson kernels $p_t(x, y)$ on space variables plays an important role in [8,9]. Recall that a strongly Lipschitz domain is a space of homogeneous type. Based on this observation and the aforementioned papers [17,18] on spaces of homogeneous type, in this paper, we define the Hardy space $H^p_A(\Omega)$ by regarding Ω as a space of homogeneous type and then establish its equivalent characterization via $N_P f$ related to the Poisson kernel and also the boundedness of the associated Riesz transforms and the fractional integrals.

To state our main results, we first recall some necessary notions and notation. Assume that Ω is a *strongly Lipschitz domain* in $\mathbb{R}^n$, that is, Ω is a proper open connected set in $\mathbb{R}^n$ whose boundary is a finite union of parts of rotated graphs of Lipschitz maps, at most one of these parts possibly unbounded; see [8]. Let $\vec{a} := (a_1, \ldots, a_n) \in L^2_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$, $0 \le V \in L^1_{\mathrm{loc}}(\Omega)$ and L_k, $k = 1, \ldots, n$, be as above. Define the *sesquilinear form* Q by

$$Q(f, g) := \sum_{k=1}^{n} \int_\Omega L_k f(x) \overline{L_k g(x)}\, dx + \int_\Omega V(x) f(x) \overline{g}(x)\, dx \tag{1.1}$$

with domain

$$\mathcal{D}(Q) := W^{1,2}_{\vec{a},V}(\Omega) := \left\{ f \in L^2(\Omega) : L_k f \in L^2(\Omega),\ k \in \{1, \ldots, n\}, \sqrt{V} f \in L^2(\Omega) \right\}.$$

In what follows, for any $f \in W^{1,2}_{\vec{a},V}(\Omega)$, we define its *norm* by

$$\|f\|_{W^{1,2}_{\vec{a},V}(\Omega)} := \|f\|_{L^2(\Omega)} + \sum_{k=1}^{n} \|L_k f\|_{L^2(\Omega)} + \|\sqrt{V} f\|_{L^2(\Omega)}.$$

It is known that Q is symmetric and closed. Let W be a *closed subset* of $W^{1,2}_{\vec{a},V}(\Omega)$. Then the *magnetic Schrödinger operator* A is defined to be the maximal-accretive operator on $L^2(\Omega)$ with largest domain $\mathcal{D}(A) \subset W$ such that, for all $f \in \mathcal{D}(A)$ and $g \in W$,

$$\langle Af, g \rangle = Q(f, g). \tag{1.2}$$

Then it is known that the magnetic Schrödinger operator A is self-adjoint; see, for example, [20, Proposition 1.24]. Formally, we write

$$Af := \sum_{k=1}^{n} L_k^* L_k f + Vf \tag{1.3}$$

or $A := -(\nabla - i\vec{a}) \cdot (\nabla - i\vec{a}) + V$. As in [8], we say that A satisfies the *Dirichlet boundary condition* (for simplicity, DBC) if $W := W^{1,2}_{\vec{a},V,0}(\Omega)$, where $W^{1,2}_{\vec{a},V,0}(\Omega)$ is the *closure of* $C_c^\infty(\Omega)$ *in* $W^{1,2}_{\vec{a},V}(\Omega)$.