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Second order Hamilton–Jacobi–Bellman equations with an unbounded operator☆

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ABSTRACT

This work is devoted to the study of a class of Hamilton–Jacobi–Bellman equations associated to an optimal control problem where the state equation is a stochastic differential inclusion with a maximal monotone operator. We show that the value function minimizing a Bolza-type cost functional is a viscosity solution of the HJB equation. The proof is based on the perturbation of the initial problem by approximating the unbounded operator. Finally, by providing a comparison principle we are able to show that the solution of the equation is unique.

1. Introduction

In this paper, we are concerned with the problem of the minimization of the cost functional of Bolza type

$$J(u) := \mathbb{E}\left[\int_0^T g(t, X_t, u_t)\,dt + h(X_T)\right]$$

subject to the following stochastic differential equation

$$dX_t + A(X_t)\,dt \ni b(t, X_t, u_t)\,dt + \sigma(t, X_t, u_t)\,dW_t, \quad t \in [0, T],$$

where A is a maximal monotone operator on an Euclidean space.

Existence and uniqueness for multivalued SDEs of this type have been considered for the first time by Bensoussan and Răşcanu [1] in the case of stochastic variational inequalities, where A is the subdifferential of a proper convex lower semi-continuous function. These results were extended by Cépa [2] for a general maximal monotone operator in a finite-dimensional setting; his approach consisted in using compacity methods within the Meyer–Zheng topology and time change techniques. Cépa came back upon the subject in [3], where he proved existence and uniqueness for the multivalued SDEs by a fixed point method in the space $L^2(\Omega; C([0, T]; \mathbb{R}^d))$. In [4], Răşcanu deals with SDEs with general maximal monotone operators defined on Hilbert spaces.

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The method which we consider uses the Hamilton–Jacobi–Bellman equation approach. We will prove that the value function of this problem is the unique solution (in the viscosity sense) of the following second order partial differential equation:

$$\begin{cases} \dfrac{\partial v}{\partial t} + \inf\limits_{u\in U} \mathcal{H}(t,x,u,Dv,D^2v) \in \langle A(x), Dv\rangle & \text{in } (0,T)\times\overline{\operatorname{Dom} A}, \\ v(T,\cdot) = h & \text{on } \overline{\operatorname{Dom} A}, \end{cases} \tag{1}$$

where the Hamiltonian $\mathcal{H} : [0,T]\times\mathbb{R}^d\times U\times\mathbb{R}^d\times\mathbb{R}^{d\times d}\to\mathbb{R}$ is defined by

$$\mathcal{H}(t,x,u,q,Q) := \frac{1}{2}\operatorname{tr}\sigma\sigma^*(t,x,u)Q + \langle b(t,x,u),q\rangle + g(t,x,u).$$

Such equations have already been studied for the first order case in [5,6]. The results obtained here were announced in [7] for the case in which the state equation is a stochastic variational inequality.

The existence of a solution of the associated Hamilton–Jacobi–Bellman equation is often proven via the dynamic programming principle via Itô calculus. However, we will obtain both the existence result and the dynamic programming principle independently, by a penalization method. We will actually employ the Yosida approximation of the unbounded operator A in order to use results already stated for controlled SDEs with Lipschitz coefficients (see [8], for example).

The uniqueness of the solutions of Eq. (1) is shown using the tools developed in [9]; the difficulties arising from the presence of the unbounded operator A are dealt with the help of its monotonicity.

2. Preliminaries

Let us recall the definition and some properties of maximal monotone operators. The interested reader can find a general development on this topic in [10], for example.

A *multivalued operator* on $\mathbb{R}^d$ is a map $A:\mathbb{R}^d\to 2^{\mathbb{R}^d}$ (or shortly, $A:\mathbb{R}^d\rightrightarrows\mathbb{R}^d$), with $2^{\mathbb{R}^d}$ denoting the class of all subsets of $\mathbb{R}^d$. Such an operator can be identified with its *graph*:

$$\operatorname{Gr} A := \left\{\left(x,x^*\right)\in\mathbb{R}^d\times\mathbb{R}^d \mid x^*\in A(x)\right\}.$$

We also define the *domain* of A:

$$\operatorname{Dom} A := \left\{x\in\mathbb{R}^d \mid A(x)\neq\emptyset\right\}.$$

A *monotone operator* on $\mathbb{R}^d$ is a multivalued operator A such that

$$\left\langle x-y, x^*-y^*\right\rangle \geq 0,\quad \forall\left(x,x^*\right),\left(y,y^*\right)\in\operatorname{Gr} A,$$

where $\langle\cdot,\cdot\rangle$ is the usual scalar product in $\mathbb{R}^d$.

The class of monotone operators on $\mathbb{R}^d$ can be ordered by the inclusion between their graphs. A maximal element with respect to this order is called a *maximal monotone operator* on $\mathbb{R}^d$. If A is a maximal monotone operator on $\mathbb{R}^d$, then $\operatorname{int}(\operatorname{Dom} A)$, $\overline{\operatorname{Dom} A}$ are convex sets and

$$\operatorname{int}(\operatorname{Dom} A) = \operatorname{int}(\overline{\operatorname{Dom} A}).$$

Moreover, for every $x\in\mathbb{R}^d$, $A(x)$ is a closed convex subset of $\mathbb{R}^d$.

An essential property of a maximal monotone operator A is the local boundedness of A on $\operatorname{int}(\operatorname{Dom} A)$, *i.e.* $\bigcup_{x\in K} A(x)$ is bounded for every compact set $K\subseteq\operatorname{int}(\operatorname{Dom} A)$.

If A is a maximal monotone operator and $\varepsilon>0$, then $(I+\varepsilon A)^{-1}$ is a Lipschitz function on $\mathbb{R}^d$ with Lipschitz constant equal to 1. This function will be denoted by J_ε. We also define the *Yosida approximation* of the operator A:

$$A_\varepsilon(x) := \frac{1}{\varepsilon}\left(x - J_\varepsilon(x)\right),\quad x\in\mathbb{R}^d.$$

An important property of these functions is that for every $x\in\mathbb{R}^d$ we have

$$A_\varepsilon(x)\in A\left(J_\varepsilon(x)\right)$$

and

$$\lim_{\varepsilon\to 0} J_\varepsilon(x) = \operatorname{Pr}_{\overline{\operatorname{Dom} A}}(x).$$

A most remarkable example of a maximal monotone operator is the subdifferential of a proper, convex, lower semicontinuous (l.s.c.) function $\varphi:\mathbb{R}^d\to\,]-\infty,+\infty]$, defined by

$$\partial\varphi(x) := \left\{x^*\in\mathbb{R}^d \mid \left\langle y-x,x^*\right\rangle + \varphi(x)\leq\varphi(y),\ \forall y\in\mathbb{R}^d\right\}.$$

If $d=1$, every maximal monotone operator on $\mathbb{R}^d$ can be expressed in this way.