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The existence of solutions to second-order singular boundary value problems

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ABSTRACT

This article analyzes qualitative properties of solutions to two-point boundary value problems for singular ordinary differential equations. In particular, we form new approaches that ensure that all possible solutions satisfy certain *a priori* bounds. The methods involve differential inequalities.

The ideas are then applied to generate new existence results for solutions to the boundary value problems under consideration. Many of the results are novel for both the singular and the nonsingular cases.

The main concepts are further illustrated via applications to various equations arising in modeling, such as the Poisson–Boltzmann equation, and the Thomas–Fermi equation with an ionized atom.

1. Introduction

This article generates new *a priori* bounds on the solutions to the second-order singular differential equation

$$\frac{1}{p}(p\mathbf{y}')' = q\mathbf{f}(t, \mathbf{y}), \quad 0 < t < T \tag{1.1}$$

coupled with various forms of the following boundary conditions:

$$-\alpha\mathbf{y}(0) + \beta \lim_{t\to 0^+} p(t)\mathbf{y}'(t) = \mathbf{c} \tag{1.2}$$

$$\gamma\mathbf{y}(T) + \delta \lim_{t\to T^-} p(t)\mathbf{y}'(t) = \mathbf{d}. \tag{1.3}$$

The new *a priori* bounds are obtained via the formulation and application of differential inequalities. The results are then further applied to the singular boundary value problem (BVP) (1.1)–(1.3) by furnishing new sufficient conditions under which at least one solution will exist. All results are new for the singular case, and are novel for the nonsingular ($p \equiv 1 \equiv q$) situation for the case of Dirichlet–Sturm–Liouville boundary conditions ($\alpha/\beta > 0$ or $\gamma/\delta > 0$). In addition, our results are new for the vector and scalar cases.

Above, $\mathbf{f} \in C([0, T] \times \mathbb{R}^n; \mathbb{R}^n)$, that is, $\mathbf{f}$ is a continuous function from $[0, T] \times \mathbb{R}^n$ to $\mathbb{R}^n$. In addition,

$$p \in C([0, T]; \mathbb{R}) \cap C^1((0, T); \mathbb{R}) \quad \text{with } p > 0 \text{ on } (0, T) \tag{1.4}$$

$$q \in C((0, T); \mathbb{R}) \quad \text{with } q > 0 \text{ on } (0, T). \tag{1.5}$$

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The application of singular differential equations to the modeling of dynamical phenomena naturally motivates the need for a deeper understanding of the qualitative aspects of solutions, including existence results. For example, the Poisson–Boltzmann equation

$$y'' + \frac{\omega}{t}y' = f(t, y), \quad 0 < t < 1 \tag{1.6}$$

$$\lim_{t\to 0^+} y'(t) = y(1) = 0, \quad \omega \geq 1 \tag{1.7}$$

arises in the theory of thermal explosions (see Chambre [1]) and in the theory of electrohydrodynamics (see Keller [2]).

By a solution to (1.1) we mean a function $\mathbf{y} \in C([0, T]; \mathbb{R}^n) \cap C^2((0, T); \mathbb{R}^n)$ with $p\mathbf{y}' \in C([0, T]; \mathbb{R}^n)$ and $\mathbf{y}$ satisfying (1.1) on $(0, T)$.

Much of the classical research on the existence of solutions to boundary value problems featuring singular differential equations can be found in the monographs [3] and in [4]. Therein, a common technique for treating the problem of the existence of solutions is to reduce the challenge to that of obtaining *a priori* bounds on solutions to a certain family of singular BVPs. In fact, it is not difficult to find abstract theorems in the literature that *assume* the existence of these *a priori* bounds; for example, see O'Regan [5,6], [3, Chap. 3]. The ideas of topological transversality of Granas et al. [7] (or other topological methods) can then be applied to appropriate operator equations to obtain the existence of solutions.

If topological methods are to be successfully applied to singular BVPs then, from a practical point of view, there is a need for concrete, easily verifiable conditions that ensure that the desired *a priori* bounds on solutions to the singular BVPs under consideration actually do exist. This paper is designed to meet this need.

In Section 2 the main ideas are presented, with the new results underpinned by two wide-ranging differential inequalities on $\mathbf{f}$. The significance and applicability of these inequalities are further illustrated in Section 3, where they are applied to various equations from modeling, including the Poisson–Boltzmann equation, and the Thomas–Fermi equation with an ionized atom.

Our new results naturally complement recent advances in the literature, for example, [8–11], where other methods from nonlinear analysis were used, including fixed-point theory in cones, and the Leray–Schauder degree. In addition, our results extend those of Rudd and Tisdell [12] from the nonsingular to the singular case.

For $\mathbf{u} \in \mathbb{R}^n$ we define $\|\mathbf{u}\|$ via $\|\mathbf{u}\|^2 = \langle \mathbf{u}, \mathbf{u}\rangle$, where $\langle\cdot,\cdot\rangle$ is the usual Euclidean dot product of two vectors in $\mathbb{R}^n$.

Our methods for forming *a priori* bounds on solutions to (1.1)–(1.3) involve differential inequalities constructed from Lyapunov-type functions and are motivated by the work of Hartman [13, p. 433], [14], Mawhin [15], Rudd and Tisdell [12], and Erbe et al. [16]. Let $\mathbf{y} = \mathbf{y}(t)$ be a solution to (1.1) and define the Lyapunov-type function $r(t) := \|\mathbf{y}(t)\|^2$. For all $t \in (0, T)$ we have

$$r'(t) = 2\langle \mathbf{y}(t), \mathbf{y}'(t)\rangle; \tag{1.8}$$

and

$$\begin{aligned}(p(t)r'(t))' &= 2\left[\langle \mathbf{y}(t), p(t)\mathbf{y}'(t)\rangle\right]' \\ &= 2\left[\langle \mathbf{y}(t), (p(t)\mathbf{y}'(t))'\rangle + p(t)\|\mathbf{y}'(t)\|^2\right] \\ &= 2\left[\langle \mathbf{y}(t), p(t)q(t)\mathbf{f}(t, \mathbf{y}(t))\rangle + p(t)\|\mathbf{y}'(t)\|^2\right].\end{aligned} \tag{1.9}$$

The above identities will be needed in the proofs of our main results.

2. The main results

2.1. Dirichlet conditions

We now discuss (1.1) subject to a special case of the boundary conditions (1.2), (1.3), namely,

$$\mathbf{y}(0) = \mathbf{c} \tag{2.1}$$

$$\mathbf{y}(T) = \mathbf{0}. \tag{2.2}$$

The above boundary conditions are a special form of Dirichlet conditions (involving $\mathbf{y}$ at the end points). Beginning with this simpler form of boundary conditions means that the flavor of the basic ideas of this work can more easily be conveyed to the reader. Furthermore, the boundary conditions (2.1), (2.2) will come into play in later sections when we address the Thomas–Fermi equation.

Theorem 2.1. *Let* $\mathbf{f} \in C([0, T] \times \mathbb{R}^n; \mathbb{R}^n)$ *and let* (1.4) *and* (1.5) *hold with*

$$K_0 := \int_0^T \frac{ds}{p(s)} < \infty, \tag{2.3}$$

$$K_1 := \int_0^T \frac{1}{p(s)} \int_s^T p(x)q(x)\, dx\, ds < \infty. \tag{2.4}$$