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On weak*-extensible Banach spaces

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ABSTRACT

We study the stability properties of the class of weak*-extensible spaces introduced by Wang, Zhao, and Qiang showing, among other things, that weak*-extensibility is equivalent to having a weak*-sequentially continuous dual ball (in short, w*SC) for duals of separable spaces or twisted sums of w*SC spaces. This shows that weak*-extensibility is not a 3-space property, solving a question posed by Wang, Zhao, and Qiang. We also introduce a restricted form of weak*-extensibility, called separable weak*-extensibility, and show that separably weak*-extensible Banach spaces have the Gelfand–Phillips property, although they are not necessarily w*SC spaces.

1. Introduction and preliminaries

In a recent paper [1], Wang, Zhao, and Qian introduce and study the notion of a weak*-extensible Banach space as follows.

Definition 1. A Banach space X is said to be weak*-extensible if, for every subspace Y, every weak*-null sequence $(f_n) \subset Y^*$ admits a subsequence (f_m) which can be extended to a weak*-null sequence $(F_m) \subset X^*$.

The main results presented in [1] are as follows.

1. (Thm. 4.2) Every Banach space X whose dual unit ball is weak*-sequentially compact (w*SC property, in short) is weak*-extensible.
2. (Thm.4.5) The product of a weak*-extensible space by a finite dimensional space is weak*-extensible.
3. (Thm. 4.3) Subspaces and quotients of weak*-extensible spaces are weak*-extensible.

Moreover, two problems are posed in [1, remark 4.7]; recall that a property P is said to be a 3-space property if a space X containing a subspace Y such that both Y and X/Y have P must also have P.

(4) Is weak*-extensibility a 3-space property?
(5) Does a weak*-extensible space have a weak*-sequentially compact dual ball?

In this paper, we introduce new techniques to study weak*-extensible spaces which allow us to obtain generalizations for results (1)–(3), partial answers to (5), and a negative answer to problem (4).

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Definition 2. Given a subspace Y of a Banach space X, we say that Y is weak*-extensible in X if any weak*-null sequence $(f_n) \subset Y^*$ admits a subsequence (f_m) which can be extended to a weak*-null sequence $(F_m) \subset X^*$.

This notion was introduced in [2]. Recall that a weak*-null sequence (f_n) on the dual of a Banach space Z can be identified with the operator $Z \to c_0$ defined by $x \to (f_n(x))_n$. It is obvious that, if Y is complemented in Z, then it is weak*-extensible in Z. Sobczyk's theorem is usually stated as follows: if Y is a subspace of a separable space Z, then every operator $Y \to c_0$ can be extended to an operator $Z \to c_0$. However, it follows from Sobczyk's proof that, if Y is a subspace of a space Z in such a way that Z/Y is separable, then every operator $Y \to c_0$ can be extended to an operator $Z \to c_0$. Thus, every subspace Y of Z such that Z/Y is separable is weak*-extensible. It is equally obvious that, given $Y \subset W \subset Z$, if Y is weak*-extensible in W and W is weak*-extensible in Z, then Y is weak*-extensible in Z. If c_0 is weak*-extensible in Z, then a subsequence of the canonical basis must span a complemented subspace. Hence, no copy of c_0 is weak*-extensible in ℓ_∞ [1, Cor.3.3]. A short exact sequence of Banach spaces is a diagram

$$0 \longrightarrow Y \xrightarrow{\ j\ } X \xrightarrow{\ q\ } Z \longrightarrow 0$$

formed by Banach spaces and linear continuous operators such that the image of each arrow coincides with the kernel of the next one. The middle space X is also called a *twisted sum* of Y and Z. It follows from the definition that j embeds Y as a subspace of X and, thanks to the open mapping theorem, Z is isomorphic to $X/j(Y)$. The sequence is said to be trivial, or to split, if $j(Y)$ is complemented in X, in which case X is isomorphic to $Y \oplus Z$.

2. Separably weak*-extensible spaces

Let us introduce a useful variation of the notion of weak*-extensibility.

Definition 3. A Banach space X is said to be separably weak*-extensible if any closed separable subspace Y is weak*-extensible in X.

An interesting link can be established with the separable complementation property (in short, $\mathcal{SCP}$); recall from [3], see also [4], that a Banach space X is said to have the separable complementation property if every separable subspace of X is contained in a separable subspace complemented in X. The following is blatantly obvious.

Lemma 1. *Spaces with the $\mathcal{SCP}$ are separably weak*-extensible.*

The converse fails. There are spaces having a weak*-sequentially compact dual ball which fail the $\mathcal{SCP}$. The simplest example is perhaps provided by any non-trivial exact sequence

$$0 \longrightarrow c_0 \longrightarrow X \longrightarrow c_0(\mathfrak{c}) \longrightarrow 0,$$

where $\mathfrak{c}$ is the cardinal of the continuum. They exist, as can be seen in [5,6]. On the one hand, by virtue of Sobczyk's theorem, any space containing an uncomplemented copy of c_0 must fail the $\mathcal{SCP}$; hence X does not have the $\mathcal{SCP}$. On the other hand, it is easy to see that $X^* \simeq \ell_1(\mathfrak{c})$; hence X^* has density character $\mathfrak{c}$, and therefore X^* has cardinal $\mathfrak{c}$, as well as its unit ball. The Cech–Pospisil theorem [7] establishes that any compact not sequentially compact space must have cardinal at least $2^{\aleph_1}$; therefore, under the continuum hypothesis, in short CH, the space X is w*CS. The list of known spaces with the $\mathcal{SCP}$ includes weakly compactly generated spaces; in particular, reflexive spaces (see [8]); weakly sequentially complete Banach lattices; in particular, $\ell_1(\Gamma)$-spaces [9]; Banach spaces with the commuting bounded approximation property [10]; duals of Asplund spaces (see [11, p.38]; see also [12, Thm. 3.42]); Plichko spaces (see [13]); there also exist non-Plichko $C(K)$-spaces with the $\mathcal{SCP}$ (see [14]). We add a new example to this list.

Proposition 1. *For every ordinal α, the space $C(\alpha)$ has the $\mathcal{SCP}$ and is weak*-extensible.*

The proof combines [15, Thm. 1.6], where Kalenda and Kubiś show that $C(\alpha)$ spaces for any ordinal α enjoy the so-called *controlled separable complementation property* (in short, $\mathcal{CSCP}$) with [16, Thm. 1.1], where Ferrer and Wójtowicz show that the $\mathcal{CSCP}$ implies w*SC.

A Banach space X is said to have the Gelfand–Phillips property if limited sets are relatively compact. Recall that a set A is said to be limited if, whenever (f_n) is a weak*-null sequence in X^*, one has

$$\lim_n \sup_{a \in A} |f_n(a)| = 0.$$

In [17, Cor. 1.3.3], Schlumprecht characterizes the Gelfand–Phillips property in $C(K)$-spaces as follows: every sequence in X equivalent to the canonical basis of c_0 admits a subsequence spanning a complemented subspace. This immediately yields that separably weak*-extensible $C(K)$-spaces have the Gelfand–Phillips property. We show that this result holds in arbitrary spaces. We start with a characterization of the Gelfand–Phillips property surely known but for which we have found no specific reference in the literature.