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L_p-theory for a generalized nonlinear viscoelastic fluid model of differential type in various domains

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ABSTRACT

This paper studies a coupled system of hyperbolic and parabolic equations governing the motion of viscoelastic, incompressible fluids on various domains. The model under consideration covers a wide range of nonlinear fluids including generalized Newtonian fluids, generalized Oldroyd-B fluids or Peterlin approximations. Existence and uniqueness of strong L_p-solutions for large times are proved for small data, for arbitrarily large data local well-posedness is shown.

1. Introduction

A commonly used model to describe the motion of a fluid in mathematical and engineering applications is the Navier–Stokes system which is based on a linear dependence of extra stress on the deformation tensor. In many cases, however, more complex fluids, such as polymeric liquids, biological fluids, suspensions or liquid crystals, exhibit behavior which cannot be characterized by this relation alone. Shear-thinning (or respectively shear-thickening), stress-relaxation, nonlinear creeping and more observed effects call for more general models.

In this work, we consider a nonlinear model describing the motion of an incompressible viscoelastic fluid given by the following quasilinear system of coupled hyperbolic and parabolic partial differential equations:

$$\begin{cases} \rho(\partial_t u + u.\nabla u) - \operatorname{div} S_v(Du) + \nabla\pi = \operatorname{div}\mu(\tau) + f & \text{in } (0,T_0)\times\Omega, \\ \operatorname{div} u = 0 & \text{in } (0,T_0)\times\Omega, \\ \partial_t\tau + u.\nabla\tau + b\tau = g(\nabla u,\tau) & \text{in } (0,T_0)\times\Omega, \\ u|_{\partial\Omega} = 0 & \text{on } (0,T_0)\times\partial\Omega, \\ u(0) = u_0 & \text{in } \Omega, \\ \tau(0) = \tau_0 & \text{in } \Omega. \end{cases} \tag{1.1}$$

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Here, the unknowns are the velocity field u, the pressure π and the elastic part of the stress τ. Furthermore, ρ is the density, $S_v(Du)$ the viscous part of the stress tensor, f exterior body force and μ and g some given functions. The deformation tensor is denoted by $Du = \frac{1}{2}(\nabla u + \nabla u^T)$.

For the viscous part of the stress, we impose the generalized Newtonian law

$$S_v(Du) = 2\alpha_1(\mathrm{tr}((Du)^2), \mathrm{tr}((Du)^3))Du + 2\alpha_2(\mathrm{tr}((Du)^2), \mathrm{tr}((Du)^3))(Du)^2 \tag{1.2}$$

where α_1 is the viscosity function and α_2 relates to the cross-viscosity. Note that for divergence free functions $\mathrm{tr}((Du)^2) = |Du|^2$ and $\mathrm{tr}((Du)^3) = \det Du$.

Roughly speaking, we prove existence and uniqueness of a strong solution (u, π, τ) up to an arbitrary time $T_0 > 0$ for a wide class of domains provided the data is sufficiently small, $g(0, 0) = 0$ (or smallness of $|g(0, 0)|$ if the domain Ω is bounded) and $\alpha_1(0, 0) > 0$. Note that – besides regularity – no further restrictions on the structure of μ, α_1, α_2 and g are imposed. Moreover, for constant $\alpha_1 > 0$ and $\alpha_2 = 0$ local strong well-posedness is proved for arbitrarily large data in the same setting. In this case, the condition $g(0, 0) = 0$ can be omitted if Ω is bounded.

Before we state our main result precisely, let us briefly discuss related results found in the literature.

In [1], strong L_p-theory for generalized Newtonian fluids without elasticity, i.e. incompressible viscous fluids satisfying the generalized Newtonian law (1.2) and $\tau \equiv 0$, is considered for small initial data u_0. Recently, Bothe and Prüß [2] have extended these results to the case of large initial data.

The special case of Oldroyd-B fluids (cf. [3]), i.e. for constant $\alpha_1 > 0, \alpha_2 = 0$ and setting

$$\mu(\tau) = \tau \quad \text{and} \quad g(\nabla u, \tau) = \beta Du - \tau Wu + Wu\tau + a(Du\tau + \tau Du) \tag{1.3}$$

for $\beta > 0, -1 \le a \le 1$ and $Wu = \frac{1}{2}(\nabla u - \nabla u^T)$, has been investigated by Guillaupé and Saut in the L_2-setting [4] in bounded domains. They proved the existence of local strong solutions for large data as well as global solutions for small data with a Schauder fixed point argument. Their method relies on a priori estimates and compactness arguments.

Later, Fernández-Cara et al. [5] proved the existence of unique strong solutions in an L_p-setting similar to our approach for the same model problem as Guillaupé and Saut also in a bounded domain. They rely on a Schauder fixed point argument as well.

A more general system than Oldroyd-B, where in (1.3) the constant term β is replaced by a shear-rate dependent function $\beta(|Du|^2)$, has been investigated in the steady L_2-setting on bounded and exterior domains by Arada and Sequeira [6,7]. This model is called generalized Oldroyd-B.

Another generalization of the Oldroyd-B model is the so-called White–Metzner system (cf. [3]), where one takes constant $\alpha_1 > 0, \alpha_2 = 0, b = 0$, the identity $\mu(\tau) = \tau$ and

$$g(\nabla u, \tau) = \beta(|Du|^2)Du + \gamma(|Du|^2)\tau - \tau Wu + Wu\tau + a(Du\tau + \tau Du)$$

for some functions β and γ. Strong well-posedness of this model in 2D has been shown in the L_2-setting by Hakim [8] and later also in 3D by Molinet and Talhouk [9] in the non-stationary case in bounded domains.

Note that, in particular, our main result shows well-posedness in L_p in all cases mentioned above.

Bringing together a nonlinear viscosity function and elastic effects, Agranovich and Sobolevskii [10] and Dmitrienko et al. [11] studied a viscoelastic fluid model in the L_2-setting on a bounded domain. However, they replaced the frame-invariant objective derivative

$$\frac{\mathcal{D}_a \tau}{\mathcal{D}t} = \partial_t \tau + u.\nabla\tau + \tau Wu - Wu\tau - a(Du\tau + \tau Du)$$

by a partial derivative ∂_t. This way, one can directly integrate the transport equation and insert the resulting elastic stress into the fluid equation.

Finally, we would like to mention a work by Vorotnikov and Zvyagin [12] who considered a problem very similar to the one we consider in this article. They proved the existence of strong solutions in the L_2-setting where $\Omega = \mathbb{R}^n$, $n = 2, 3$. However, due to the L_2-approach using a priori estimates for a nonlinear system, they impose strong regularity assumptions on the initial data, i.e. $u_0 \in H_2^3(\mathbb{R}^n)$ and $\tau_0 \in H_2^3(\mathbb{R}^n)$. It is thus likely that a generalization of their method to domains – if possible – would lead to additional compatibility conditions on the initial stress.

In contrast to their approach, our proof relies on L_p-theory and thus allows to deal with a wide class of domains without further compatibility conditions and lower regularity assumptions on the initial data.

Throughout this paper for $p, q \in [1, \infty]$, $s \in \mathbb{R}_+$ and $k \in \mathbb{N}_0$ the spaces $H_q^s(\Omega)$, $W_q^s(\Omega)$, $B_{pq}^s(\Omega)$ and $\widehat{H}_q^k(\Omega)$ denote the usual Bessel potential spaces, Sobolev spaces, Besov spaces and homogeneous Sobolev spaces, respectively. If not said otherwise, $\Omega \subset \mathbb{R}^n$ is a domain with a uniform C^2-boundary (boundary regularity is to be understood in the sense of [13, Definition 4.10]). This guarantees the existence of a total extension operator for Ω, see [13, Theorem 5.24]. Hence, interpolation theory for Bessel potential spaces and Besov spaces, Sobolev embedding theorems and the mixed derivative theorem are valid in Ω.

Moreover, for $q \in (1, \infty)$ and a domain $\Omega \subset \mathbb{R}^n$ we define the space of solenoidal vector fields by

$$L_{q,\sigma}(\Omega) = \overline{\{u \in C_c^\infty(\Omega)^n : \operatorname{div} u = 0\}}^{L_q(\Omega)}.$$