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Vanishing theorems on hypersurfaces in Riemannian manifolds☆

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ABSTRACT

We study a complete noncompact stable minimal hypersurface M and a strongly stable hypersurface M with constant mean curvature in a 5-dimensional Riemannian manifold N. If N is a compact simply connected manifold with bounded sectional curvature $\frac{5}{17} \leq \bar{K} \leq 1$, then there is no nontrivial L^2 harmonic form on M. This is a generalized version of Tanno's result on a stable minimal hypersurface in $\mathbb{R}^5$ and Zhu's result on a stable minimal hypersurface in $\mathbb{S}^5$.

1. Introduction

Each complete minimal graph in $\mathbb{R}^{n+1}$ $(n \leq 7)$ is a hypersurface. If $n \geq 8$, then it is false. Do Carmo and Peng [1] showed that complete orientable stable minimal surfaces in $\mathbb{R}^3$ are planes. At the same time, Fischer-Colbrie and Schoen [2] independently showed that a complete stable minimal hypersurface M in a complete 3-dimensional manifold N with nonnegative scalar curvature must be either conformally a plane or conformally a cylinder $\mathbb{R} \times \mathbb{S}^1$. For the special case, they also prove that M must be a plane if $N = \mathbb{R}^3$. Palmer [3] proved that there is no non-trivial L^2 harmonic 1-form on a complete noncompact orientable stable minimal hypersurface in $\mathbb{R}^{n+1}$. It implies that there exists some topological obstruction for the stability of M. Tanno [4] showed that there is no nontrivial L^2 harmonic 1-form on a complete noncompact orientable minimal hypersurface in N^{n+1} with non-negative bi-Ricci curvature. Cheng [5] confirmed this result to hold for a complete noncompact orientable strongly stable hypersurface M with constant mean curvature H in an ambient manifold N^{n+1} with bi-Ricci curvature having a low bound $\frac{n(n-5)}{4}H^2$ along M. Tanno [4] proved that if M is a complete orientable stable minimal hypersurface in $\mathbb{R}^5$ then there exist no non-trivial L^2 harmonic p-forms on M $(0 \leq p \leq 4)$. The author [6] proved that a complete noncompact orientable stable minimal hypersurface in $\mathbb{S}^5$ admits no nontrivial L^2 harmonic forms and also obtained that a complete noncompact strongly stable hypersurface with constant mean curvature in $\mathbb{R}^5$ or $\mathbb{S}^5$ admits no nontrivial L^2 harmonic forms.

In this paper, we consider L^2 harmonic forms on a (strongly) stable hypersurface M^4 in a 5-dimensional manifold N^5 whose sectional curvature is bounded. First, we fix some notations so as to state the main result. Suppose that N^5 is a 5-dimensional Riemannian manifold and $x : M^4 \to N^5$ is an isometric immersion of a 4-dimensional orientable manifold M with constant mean curvature H. We denote $\bar{R}$, $\overline{\text{Ric}}$, $\bar{K}$, R, Ric and K by the curvature tensor, the Ricci curvature, the sectional curvature of N and the curvature tensor, the Ricci curvature, the sectional curvature of M, respectively. ∇ denotes the Levi-Civita connection of M. γ is the unit normal vector field of M. $|A|$ is the normal of the second fundamental form $A = (h_{ij})$ of x.

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In the minimal hypersurface case, the immersion x is called stable if

$$I(h) = \int_M |\nabla h|^2 - (\overline{\text{Ric}}(\gamma, \gamma) + |A|^2)h^2 dv \geq 0, \tag{1.1}$$

for all compactly supported piecewise smooth functions h on M, where ∇h is the gradient of h and dv is the volume form. In the case of nonzero constant mean curvature, the immersion x is called strongly stable if (1.1) holds for all compactly supported piecewise smooth functions h on M. The Hodge operator $* : \wedge^p(M) \to \wedge^{4-p}(M)$ is defined by

$$*e^{i_1} \wedge \cdots \wedge e^{i_p} = \text{sgn}\sigma(i_1, i_2, i_3, i_4) e^{i_{p+1}} \wedge \cdots \wedge e^{i_4},$$

where $\sigma(i_1, i_2, i_3, i_4)$ denotes a permutation of the set (i_1, i_2, i_3, i_4) and sgnσ is the sign of σ. The operator $d^* : \wedge^p(M) \to \wedge^{p-1}(M)$ is given by

$$d^*\omega = - * d * \omega.$$

The Laplacian operator is defined by

$$\Delta\omega = -dd^*\omega - d^*d\omega.$$

A p-form ω is called L^2 harmonic if $\Delta\omega = 0$ and

$$\int_M \omega \wedge *\omega < +\infty.$$

Denote by $H^p(L^2(M))$ the space of all L^2 harmonic p-forms. We obtain that Tanno's result still holds when the ambient manifold is a 5-dimensional manifold with bounded sectional curvature. More precisely, we obtain the following result.

Theorem 1.1. *Let M^4 be a complete noncompact orientable stable minimal hypersurface or a complete noncompact orientable strongly stable hypersurface with constant mean curvature in a Riemannian manifold N^5. If N^5 is a compact simple connected manifold whose sectional curvature $\bar{K}$ satisfies $\frac{5}{17} \leq \bar{K} \leq 1$, then $H^p(L^2(M)) = \{0\}$, for $0 \leq p \leq 4$.*

Remark 1.2. From the viewpoint of topology, by pinched theorem, N^5 is only homeomorphism to S^5 in Theorem 1.1. But, N^5 may have different differential structures.

2. Proof of main results

We initially introduce two algebraic results which will be used later.

Lemma 2.1 ([7]). *Let (V^m, g) be an m-dimensional vector space with the metric g. Let $\Omega \in \wedge^2 V^*$. There exists an orthonormal basis $\{e_1, \ldots, e_{2n}, \ldots, e_m\}$ (its dual is $\{e^1, \ldots, e^{2n}, \ldots, e^m\}$) such that*

$$\Omega = \sum_{i=1}^{n} \alpha_i e^{2i-1} \wedge e^{2i},$$

where $2n \leq m$.

Lemma 2.2 ([6]). *Suppose that A is a symmetric 4×4 matrix and B is an antisymmetric 4×4 matrix. Then, we have the following relation:*

$$-2\text{tr}A \cdot \text{tr}AB^2 + 2\text{tr}A^2B^2 + 2\text{tr}(AB)^2 + |A|^2|B|^2 \geq 0.$$

Let (M, g) be a 4-dimensional Riemannian manifold. Let $\{e_1, e_2, e_3, e_4\}$ be locally defined orthogonal frame fields of tangent bundle TM. We denote $\{e^1, e^2, e^3, e^4\}$ by the dual coframe fields. Suppose $\omega = a_{i_1 i_2} e^{i_2} \wedge e^{i_1} = a_I \omega_I$, and $\theta = b_{i_1 i_2} e^{i_2} \wedge e^{i_1} = b_I \omega_I$, where the summation is being performed over the multi-index $I = (i_1, i_2)$, $a_{i_1 i_2} = -a_{i_2 i_l}$ and $b_{i_1 i_2} = -b_{i_2 i_1}$. Set

$$\langle \omega, \theta \rangle = \sum_I a_I b_I$$

and

$$|\nabla\omega|^2 = \sum_{i=1}^{4} |\nabla_{e_i}\omega|^2.$$

It is known that the following equality always holds for each 2-form ω [8]:

$$\Delta|\omega|^2 = 2\langle \Delta\omega, \omega \rangle + 2|\nabla\omega|^2 + 2\langle E(\omega), \omega \rangle, \tag{2.1}$$